The 9th MATHEMATICAL CREATIVITY AND GIFTEDNESS International Conference

2015, June, 25-28

PROCEEDINGS

MCG International Conference 2015

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The International Group for

Mathematical Creativity and Giftedness

Sinaia, ROMANIA

2015

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WELLCOMING THE MCG-9 CONFERENCE

Florence Mihaela Singer
Univ. of Ploiesti, Romania
Chair - MCG-9 Conference

Roza Leikin
University of Haifa., Israel
President MCG

Mathematical creativity and giftedness is a topic of increasing interest around the world. The International Group for Mathematical Creativity and Giftedness (MCG) brings together mathematics educators, mathematicians, researchers, and others who support the development of mathematical creativity and mathematical giftedness. The group supports teachers in their work directed at the realization of students' mathematical potential and developing mathematical creativity. We invite young researchers and practitioners to join our group under guidance and support of experts from the MCG community. We promote national and international collaboration of researchers and practitioners in the field of MCG. Our Newsletter updates MCG members about news in research and events associated with MCG.

Our conferences promote international collaboration between mathematics education researchers, mathematicians and practitioners and inform the educational community of research-based recommendations about effective ways of realization of mathematical potential and developing mathematical creativity.

The 9th edition of the International Conference *MATHEMATICAL CREATIVITY AND GIFTEDNESS* (MCG-9) is to be held in Sinaia, Romania, on June 25-28, 2015. This edition is organized by the Institute for the Development of Educational Assessment (IDEA) and the University of Ploiesti (UPG), Romania, two institutions who intend to make the conference a valuable event. In a manner that already became a tradition, the topics addressed within this edition are: mathematical creativity for all students, from all backgrounds, and of all ages; mathematical creativity, aptitude, and achievement; mathematical giftedness, talent and promise; mathematical creativity for individuals or teams, inside or outside the classroom; mathematics competitions. The conference will bring together teachers and researchers from 17 countries from 4 continents.

We benefit from the presence of well-known plenary speakers, who will provide the audience food for thought from very diverse perspectives. Although the perspectives will be diverse, they will have a common characteristic: the focus on the practicalities of applying elaborated theories into real school settings. Thus, Bharath Sriraman will discuss *Mathematical Pathologies as Pathways into Creativity*, showing how some “engineered” pathologies can stimulate creativity in the mathematics classroom. Gabriele Kaiser will approach the *Professional competence of teachers and its relation to creativity* starting from a theoretical framework and advancing to to devise the teacher’s competences that foster and support students’ creativity. Viktor Freiman will invite us to travel to a future that tends to become part of our present times: *Mathematical giftedness goes online: what are the new ways, tools, resources to develop talents and creativity in students?* Moving from exploring technology to exploring our own thinking power as teachers, Rina Zazkis proposes *Virtual conversations on mathematical tasks*, by inviting teachers to anticipate complex designs of imaginary conversations between the teacher and the students - a recent approach implemented in teacher education.

During the conference, a series of research reports and research projects will be presented, organized around the following topics: *Creativity in elementary education*; *Creativity in Mathematics Competitions*; *Problem solving and problem posing situations*; *Creativity in early years*; *Expertise versus mathematical giftedness*; *Developing creativity through challenging activities*; *Mathematical giftedness, talent and promise*; *Challenges in the teaching practice*; and *Teaching discourse in creative learning environments*.

A series of practical aspects related to giftedness and creativity will be approached within the five planned workshops: *Rational Creativity: Algorithms or Innovation? Teaching Operations with Fractions* (Linda Jensen Sheffield); *Effective feedback for efficient learning: A computer-based system of assessment* (Florence Mihaela Singer and Cristian Voica); *Visualizing geometric concepts using a novel type of 3D puzzles* (Consuela Luiza Voica and Aurelia Grigorescu); *Activities that Engage Gifted and Talented Students in Productive Struggles with Desirable Difficulties: A Model Encouraging Intense Discourse and Deep Reasoning* (William Renwick Speer) and the *Use of algebraic reasoning for early identification of mathematical talent* (Sinan Kanbir).

During the two Symposia organized within the conference, Bronislaw Czarnocha and William Baker wil discuss about *The nature of creativity*, while Marianne Nolte will interact with the audience while presenting ways for *Fostering Mathematical Giftedness in regular classroom setting and in special programs.*

On the same note of connecting theoretical frameworks to practical solutions, the Research panel organized by: Roza Leikin, Florence Mihaela Singer, Linda Sheffield, and Jong Sool Choi will stress on the *Connection between theory and practice in gifted education*.

We welcome all the participants of the 9th Conference of the International Group for Mathematcal Creativity and Giftedness and hope this will be a interesting and pleasant, fruitful and inspiring event that will lead to new collaborative projects, discoveries and friendships.

The 9th International Conference on
MATHEMATICAL CREATIVITY AND GIFTEDNESS

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Plenary lectures - ABSTRACTS

PLENARY LECTURE

MATHEMATICAL PATHOLOGIES AS PATHWAYS INTO CREATIVITY

Bharath Sriraman
University of Montana, USA
Dept of Mathematical Sciences

***Abstract**. In this plenary, we will explore mathematical pathologies as a means of fostering creativity in the classroom. In particular we will delve into what constitutes a mathematical pathology, examine classical mathematical pathologies and broaden this as a pathway into creativity. Teaching experiments with "engineered" pathologies that facilitate connections and generalizations that reflect creativity in the mathematics classroom will be presented.*

PLENARY LECTURE

PROFESSIONAL COMPETENCES OF TEACHERS AND ITS RELATION TO CREATIVITY

Gabriele Kaiser
University of Hamburg, Germany
Faculty of Education
Didactics of Mathematics

***Abstract**. The talk will depart from a theoretical framework developed in the international comparative study on future mathematics teachers, namely the "Teacher Education and Development Study in Mathematics" (TEDS-M) and its follow-up study "TEDS-FU", which focuses early career mathematics teachers. The theoretical framework departs from a model of professional competency of mathematics teachers combining cognitive and affective facets of professional knowledge of teachers with situated, practice-oriented competence facets based on noticing as central concept. The results of these studies will be presented focusing on the perspective of teachers' competencies to foster and support students' creativity and its relation to the professional competence of teachers.*

PLENARY LECTURE

MATHEMATICAL GIFTEDNESS GOES ONLINE: WHAT ARE THE NEW WAYS, TOOLS, RESOURCES TO DEVELOP TALENTS AND CREATIVITY IN STUDENTS?

Viktor Freiman
Université de Moncton, Campus Moncton
Faculté des sciences de l'éducation
Université de Moncton, Canada

Abstract. *An almost unlimited number of online resources, tools and activities are now available for everyone who has an Internet access. In the lecture, I will reflect on the impact of this phenomenon on mathematical giftedness, creativity and talent development from the point of view of mathematics educator. The Net Generation is a relatively new concept in the field of educational studies. It designates a generation of young learners that has been grown with computers, the Internet and interactive multimedia tools. Using extraordinary abilities to adapt to all new tools that are constantly arriving on the market and turn them into a specific social network, they expand their learning space beyond the traditional classroom. Blogs, wikis, web- and pod-casting are just a few examples of new ICT tools available for knowledge building, knowledge sharing and socialization. Are we, as mathematics educators, ready to meet the learning needs and adjust to the different learning styles of this generation in order to turn its natural interest and motivation into meaningful and advanced mathematics learning? While addressing specifically to mathematically gifted students, what are online opportunities for enrichment, acceleration, compacting mathematics curriculum? What challenges can be conceived beyond the academic settings thus nurturing mathematical creativity and talent? What are alternative forms of mathematical activities that can be afforded by virtual environments? How to design online environments for mathematically gifted, and what kind of activities can be organised? How to make them sustainable? What are the collaboration patterns and their impact on learning? While an important body of research reveals potentially rich challenging opportunities that are provided by technology, little is known about their effect on learning outcomes and how to integrate them in the everyday teaching practices, especially for mathematically gifted and talented learners. In my presentation, I will discuss several theoretical and practical issues related to the building, implementation, maintenance, development, and researching virtual environments that can be potentially useful for these students. Examples from a variety of ICT-enhanced projects developed in New Brunswick, Canada (CAMI website, Virtual Mathematical Marathon, Robotics-Based Learning, among others) will be given. In more general way, I will also reflect on cognitive, affective and social perspectives of the learning and teaching, and research processes that integrate web 2.0 interactive tools.*

PLENARY LECTURE

VIRTUAL CONVERSATIONS ON MATHEMATICAL TASKS

Rina Zazkis
Faculty of Education
Simon Fraser University
Burnaby
Canada

Abstract. *Scripting – or writing an imaginary conversation between a teacher and her students – is an approach recently implemented in teacher education. I will explain how this approached emerged and highlight its advantages for teachers, teacher educators and researchers. I will exemplify design and implementation of scripting tasks that support creative mathematical and pedagogical ideas.*

Research Panel - ABSTRACT

RESEARCH PANEL

CONNECTION BETWEEN THEORY AND PRACTICE IN GIFTED EDUCATION

Roza Leikin
Florence Mihaela Singer
Linda Sheffield
Jong Sool Choi

Abstract. *The goal of this panel is to initiate authentic and keen debate on the issues related to connections between research and practice in the education of mathematically talented students. We will present some examples and presents somehow contrast positions on questions: Can high achievements in school mathematics be chosen as an indicator of mathematical giftedness? Can mathematical creativity be developed in all or is it a special characteristic of mathematically gifted? How can effective teaching approaches in the education of mathematically talented students be examined? What research paradigms and methodologies are the most effective in MCG research? How are research results implemented in practice? The participants will demonstrate how different perspectives on giftedness and creativity influence research design and, moreover, research findings. On the other hand, we will argue that systematic research is essential for the advancement educational practices directed at the realization of students' intellectual potential.*

ORAL PRESENTATIONS 1.1.

Creativity in elementary education
Chair of the session: **Marianne Nolte**

NURTURING STUDENTS' CREATIVITY THROUGH TELLING MATHEMATICAL STORIES

Anna Prusak
Academic Religious Teachers' College – Shaanan, Haifa, Israel
Academic College of Education – Oranim, Tivon, Israel

Abstract. *In a project aimed to develop mathematical creativity, 46 eleventh- and twelfth-grade students were asked to compose "mathematical riddles" in which mathematical concepts or objects that they had learned about were "hidden". They then presented their riddles to their classmates, who had to discover the object/concept. The students' responses to the assignment showed that they felt that it contributed significantly to their mathematical creativity. The description of the task along with examples of the students' riddles were then presented to 32 mathematics teachers, who were asked to give their opinion of the potential such an approach might have in the development of mathematical creativity in students. This lecture will present examples of the students' stories and their reflections on their experience, along with the teachers' ideas concerning the possible contribution such an exercise might make in developing mathematical creativity in students.*

Key words: mathematical creativity; writing stories in mathematics; assessment of creativity in mathematics; reflection on creative procedure; imagination; team teaching.

INTRODUCTION

Characteristics of creative mathematical thinking include flexibility, innovation, fluency and originality (Silver, 1997). According to Feldhusen (2002), originality is the essence of creativity and its ultimate product. To support creative mathematical development, I developed a didactic tool called "writing a mathematical riddle." The riddle is to be some story that has hidden within it a mathematical object or concept being studied, for which clues are scattered throughout.

This study tested whether such an activity indeed encourages development of students' mathematical creativity.

My assumption here was that when the students wrote the riddle, they would, in fact, be "personifying" the mathematical object or concept, resulting in a process of creativity. The presentation of the riddles to their peers, asking them to point out to which mathematical objects/concepts the stories are related, was very important. Students were then asked to reflect upon their experiences, and to evaluate both the process and the product. In parallel, a group of mathematics teachers were told about the idea and shown the students' stories, and were then asked to evaluate the task's potential influence in fostering students' mathematical creativity. The teachers' and students' evaluations were then compared.

THEORETICAL BACKGROUND

Numerous studies point to the role of mathematical discourse, particularly the importance of writing about mathematics as part of the process of building mathematical concepts in learners (Borasi & Sheedy, 1990). It has been recommended to encourage the independent creation of mathematical problems, and not just to rely on problems presented in textbooks (Bibby, 2002). Silver (1997) points out that, even though creativity is often considered something only for gifted students or as a special talent, it can be used in mathematics education.

Authentic mathematical activities are closely linked to creativity (see, for example, Silver, 1997), and thus cultivating mathematical creativity in students should be one of the first goals of mathematics education (NCTM, 2000).

Researchers have emphasized the importance that the process of creating mathematical problems has for the students themselves, the goal being not just understanding the material, but also as a tool that helps the teacher understand the students' thinking process (Morgan, 1996).

In 1926, Wallas presented a model of the creative process in which there were four stages: preparation, incubation, illumination and verification. Wallace's theory is still widely cited today. Ervynck (1991) defined mathematical creativity as the ability to solve problems and/or develop structured thinking while referring to the logical-didactic nature of the area of knowledge and adapting the connections to the mathematical content. He emphasized that creative activity is not related to a familiar algorithm, and usually leads to a novel concept of the definition or to an expression of a new mathematical argument and its proof.

The literature suggests changing the standard perception regarding the importance of creativity in education in general and in schools in particular, and suggests making creativity an inseparable part of the learning and teaching processes in all aspects of subjects taught, and to regard creativity as an ability that is critical and not just "nice" (Perera, 2012).

Many studies have suggested that teachers use efficient methods to evaluate the creativity of students. So that these evaluation methods are not subjective, researchers base them on a variety of solid, defined and quantitative criteria. For example, Brookhart (2013) suggests an evaluation method that focuses on the product of the creative assignment in various areas (e.g., poetry, prose, poster, project, presentation, story) in accordance with the specific criteria, such as variety of ideas and their expression in the product, variety of sources, connection between ideas, if the product "projects" innovation, etc. Other researchers emphasize the importance of criteria such as "processing/ refinement/elaborateness or complexity, and also those that characterize the creative process: such as originality and conceptual flexibility, and claim that using these enriches the learner, who receives recognition for different aspects of the learning (see for example the examples in Klavir & Gorodetsky, 2011).

Researchers in the field of mathematical creativity who have focused on encouraging mathematical creativity in students, point out the even teachers who recognize the important of nurturing creativity in students often do not make the effort for this in school (Sriraman, 2005).

PURPOSE AND RESEARCH QUESTIONS

Purpose of the research: To investigate if writing mathematical riddles contributes to the development of mathematical creativity in students.

Research questions:

1. How do students assess the influence that writing riddles has on their mathematical creativity?

2. How do teachers assess the inherent potential that writing riddles has on developing the mathematical creativity of students?

3. Is there a difference between the teachers' and students' assessments regarding the contribution that writing mathematical riddles has on students' mathematical creativity?

Based on my previous experience, I hypothesized that viewing mathematical concepts and terms as "real entities" would allow students to become better acquainted with the ideas and, as a result, express their creativity. My assumption was that the students would recognize and appreciate the impact that the writing would have on their creativity, whereas, regarding the second question, the teachers would be ambivalent to the influence such a writing task would have on students' creativity, leading to a difference between what value students and teachers saw in the exercise.

METHODOLOGY

Participating in the study were 25 grade-eleven and 21 grade-twelve students who were studying advanced mathematics (matriculation level) in schools in northern Israel. Also participating were 32 secondary-school mathematics teachers who were told about the idea of writing riddles and were shown the stories that the students wrote.

Using leading questions, the teachers were asked to assess the potential influence that writing mathematical riddles might have on the students' learning process and on the development of their mathematical creativity. An example question: as a teacher of mathematics, would you use such activities in your class? Why or why not? If yes, how?

Research tools included the riddles written by the students, questionnaires about "Your reaction to writing riddles," the students' reflective journals, detailed interviews of a sample of six, randomly-selected students following the assignment, video recordings of the students working in groups and of their presentations to the class, and the teachers' written responses regarding their assessment of the potential that such an activity might have for students. The data were analyzed according to the rules for analyzing qualitative data.

MAIN FINDINGS

The reflections expressed by the students in the questionnaires and interviews following the task showed that that writing the stories contributed to the development of their mathematical creativity. The students referred to their product (the riddle) as an expression of a creative process. They judged the product in terms of quality and creativity. The students' self-assessment indicated four stages of creativity (similar to those of Wallas): preparation (defining the issue), incubation (putting the issue aside of a period of time), illumination (discovering new ideas that arise), and verification (instructing their peers). For example:

> I didn't come up with the story immediately. The first evening I considered different ideas, and I realized that if I hid only one element it would be too easy to find, and the story would be too short. Suddenly I thought that a cosine in a triangle is a "short portion" to a "long part," and I also

> decided that the story would take place in water. Once I had the idea, the rest came to me as I wrote the draft of the story. At first I didn't think much, but then I rewrote it three times because I wasn't satisfied and because of the "creative agony." And each time, new ideas came to me.
>
> For a few days I couldn't think of a subject for the riddle story. Then I decided that my heroes (the trigonometric functions) should be living entities, because in this case I thought the story would have a more interesting character. Then I had to think up a setting and the storyline. Why I chose the island of Bermuda? Because I wanted a place that really existed but at the same time was exotic. As the course of events slowly unfolded in my mind, I finally felt that I had created a pretty exciting storyline. It took me about an hour to write the story, and then I rewrote it four times, and each time I added to it or corrected things or made changes.

The students wrote that the task made them discover their ability to create mathematically, and they felt a sense of pride and enjoyment from their creations.

> For me, writing the mathematical riddle was fascinating and very interesting, because it gave me the opportunity to invent my own ideas that weren't restricted by anything, and the ability to imagine all I wanted too. This time, instead of having to solve something, I could make up the problem by myself, and hide concepts in it, and to solve it and to show it to my classmates.
>
> I've never written anything like this story in my life, because I'm used to solving riddles, not making them up. I wrote the riddle in rhyme. I love writing in rhyme. I got the idea the moment they gave out the problem, and I waited for that "wonderful moment" when the idea takes hold. The hard part was integrating the mathematics imaginatively into the piece, but I really enjoyed making up the story.
>
> The idea itself of this assignment was new and amazing, because there's nothing like it in our textbook or anywhere else. It's like it's on the subject, but something new and fun.
>
> For me, it was fascinating to see how I could create new heroes and new storylines. I felt this was something I could really show off.

Below is an example of one of the students' (Karen) stories. Karen's story relates to spatial geometry, and she chose the "ball" as the hero of her story. As can be seen, Karen underlined what she perceived as "keywords", aiming to help her classmates discover the hidden concept.

> I asked him to show me his sides. He refused. I believe I was too direct. I decided to start over, this time differently.
>
> "Perhaps you have a headache!" He looked around. "That's OK. Many of your friends suffer from headaches." He looked at me and said he had problems.
>
> I laughed.
>
> It is quite obvious that he has problems. It is as if someone decided intentionally to pour problems upon him and his friends.
>
> I tried to encourage him, to tell him that his problems stem from his uniqueness, that everyone says how nice and balanced he is, with no irregularities.
>
> He seemed very tired, unable to relax, moving around without stopping.
>
> I suggested to him to find a nice quiet corner to sit and relax, but he said he had no chance…. Then I had another idea: "If life is so difficult for you, why don't you take it easier? Why don't you 'round

> off some corners' here and there?" I was satisfied. I believed I had finally thought of something that might relieve his pain.
>
> But he replied, "I have already rounded them all."

The students emphasized that having to share their stories with their peers inspired them to create an understandable, imaginative story, because they wanted their peers to appreciate their efforts.

As to the responses of the mathematic teachers, they demonstrated that the teachers were unable to appreciate the full potential of such an approach. Comments included sentiments such as the following: "It seems to me that such an activity is more suitable for higher-level students and shouldn't be attempted in every group"; "This activity isn't suitable for every class. It is more suitable for classes were most of the students are committed to the general purpose and not just to the individual purpose." One teacher wrote:

A student may not necessarily fully understand the topic under study when he is writing the story, meaning that writing a specific will not inevitably lead him to have more understanding of the subject or learn more about it, no matter how creative the activity is. Also, trying to solve another student's riddle will not necessarily cause a student to understand the material.

The teachers saw the process of writing a riddle mainly as a set of algorithmic phases, without appreciating the emotional and experiential aspects. Regarding the product itself, many of the teachers held reservations and pointed out that, in their opinion, the students cannot cope with the assignment: to invent an imaginative mathematical story.

The process through which a student goes when handling such an exercise begins with a good understanding of the mathematical concept. Next, the student must think "outside the box," and do some brainstorming to come up with some possible ideas for presenting his/her subject. The brainstorming is reflected in the creative thinking, since the student must think of some conceptual basis upon which to construct the riddle. The next stage tests the students' ability to express themselves in writing, bearing in mind correct linguistic structure when presenting the required information for solving the riddle. In the final stage, the student presents her riddle for solving and this is when the level of difficulty of the riddle is measured.

The student must first consider a setting that may have some connection with the content taught: not every setting is suitable for every area of study. The student then must decide on which feature in the content area to focus on. Following the thought process, comes the construction process: the student must link the chosen content to the chosen setting, and create/compose a riddle that will be interesting and require specific mathematical processes to solve it.

Reflecting on the process of writing her riddle, one student wrote:

> I started to think differently. … What did I learn from the activity? To develop my thinking, my mind and my ability to investigate, analyse and look at a mathematical object from different aspects. I understood that mathematics is not a collection of exercises waiting for me to solve, but a whole, living world, interesting and fascinating. This world has room for individual thinking, individual expression, and creativity.
>
> The most typical reflection of the students is expressed by this quote: "I now realize that there is hidden creativity in each of us".

The results show that, contrary to the teachers' assessments, students were able to express and develop their mathematical creativity. Therefore, it can be concluded that writing riddles offers an effective tool to foster mathematical creativity.

CONTRIBUTION OF THE STUDY TO THE FIELD OF MATHEMATICAL EDUCATION

The present study strengthens the concept of using a didactic tool such as "writing riddles" on various mathematical topics and presenting them in front of their peers. The results of the teachers' appraisal show that they may not appreciate the intrinsic advantages of such an approach, and thus will be unlikely to use it in their classes. However, making them aware of the positive reactions of the students may lead teachers to change their evaluation of the idea of the writing exercise and try it in their classes.

Teachers do believe in the importance of fostering creativity in their students. So that teachers can assess the contribution that a specific approach makes in cultivating student creativity, we suggest that they try the approach before making a decision.

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SOLVING A MATHEMATICAL CREATIVITY TASK

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University of Cyprus

Abstract. *The aim of this study was to investigate the process followed by individuals during solving a mathematical creativity tasks. Therefore, 182 students of age 10-12 years old participated in individual interviews while solving a multiple solution mathematical task. Through the interviews we aimed to reveal the cognitive sub-processes that students followed while they were solving the task as well as to compare the appearance of sub-processes in students of different level of mathematical creativity. The results of the study indicated that the creative process in mathematics could be described across five non-sequential sub-processes: investigating, relating, creating, evaluating and communicating. The creative process varied among students with different degree of mathematical creativity.*

Key words: mathematical creativity, creativity process, creative activity

INTRODUCTION AND THEORETICAL CONSIDERATIONS

Researchers used the term "creative process" to describe stages, actions and behaviors that are active during the generation of an idea (Johnson & Carruthers, 2006). Having a full understanding of the characteristics and the stages of the creative process, the "secret" that leads to creative outcomes may be revealed. Since the creative process is not directly observable, several questions still remain unanswered: What makes a process creative? In what way does a creative process vary from non-creative processes?". Hence, the present paper aims to give an insight into the creative process by examining the stages emerging as an individual is working in mathematics. Furthermore, we will investigate the manner in which these stages differ in students with different degree of mathematical creativity.

Numerous studies investigated the nature of the sub-processes by proposing models underpinning the creative thinking (Lubart, 1994). One commonly accepted model, is the model proposed by Wallas (1926) which includes the stages of preparation, incubation, illumination, verification. Preparation is a conscious stage involving realization of the problem, gathering information, understanding and exploring the problematic situation (Johnson & Carruthers, 2006). During incubation no conscious work is taking place (Lubart, 2001). However the solver is subconsciously still working on the problem (Johnson & Carruthers, 2006). In the stage of illumination a promising idea is suddenly come into conscious awareness (Davis & Rimm, 2004). The phase of verification includes conscious work concerning tests, configurations and development of the ideas (Johnson & Carruthers, 2006). Beside the great admittance of the Wallas' model, some researchers are still not satisfied with it (Lubart, 2001). Guilford (1950) characterized this perspective as superficial, since it tells "almost nothing about the mental operations that actually occur" (p. 451). Moreover, contemporary theoretical models move away from a fixed sequence of sub-processes, given that linear models are insufficient to represent creative processes.

Recently, Sheffield (2009) proposed a heuristic model, which includes five non-sequential stages (investigating, relating, creating, evaluating and communicating) that may enhance individuals to become more creative during problem solving. The stage of investigation refers to an in-depth study of the available information and relevant mathematical concepts and ideas. The stage of relation is defined as the process of comparing ideas, identifying similarities and differences and combining information. In the stage of evaluation students are reflecting on the proposed solutions and confirming the success of the targets that were set in the first place. The stage of communication refers to the description and explanation of ideas and strategies. At the stage of creation, individuals find solutions or identify new ideas. For the purposes of this study, we adopted the heuristic model proposed by Sheffield (2009). Although a heuristic model usually is used to enhance learning and teaching processes, in the present study we aimed to examine whether the five stages are appeared as students solve a mathematical creativity task. Hence, indications of the process followed by individuals during solving mathematical creativity task may be revealed. We considered that the model proposed by Sheffield (2009) may provide a strong ground on examining creative process for two reasons. Firstly, the model is not a linear one; hence the transition from one stage to the other is determined by the individual. In addition to this, there is no fixed starting point of the process; instead the solver may start from any point and direct the process according to his/her needs. Secondly, the sub-processes are focused on the moment of solutions/ ideas appearance and not before or after the emergence of ideas.

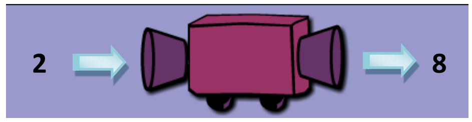
METHODOLOGY

A sample of 182 students aged 9–12 years old participated in the study (Grade 4: 53, Grade 5: 81, Grade 6: 48 students). The study was accomplished in two phases. In the *first phase*, participants asked to complete the Mathematical creativity test (MCT). Four multiple-solutions tasks with problem solving and problem posing situations included in the test and mathematical creativity was assessed across fluency, flexibility and originality. Students' performance in the MCT lead to their categorization in four groups: Group 1 (the lowest 15% of the performances), Group 2 (15% - 50% of the performances), Group 3 (50% - 85% of the performances) and Group 4 (highest 15% of the performances). In the *second phase* of the study, participants were individually asked to provide within 20 minutes as many, different and original solutions they could for a multiple solutions task (Figure 1). Using a think-aloud methodology, participants explained the proposed solutions and procedure followed. Interviews were recorded. Furthermore, students' leaflets were used for the analysis of the proposed solutions. Data were analyzed qualitatively and quantitatively. The data from the first phase of the study were analyzed using the statistical programme SPSS, whereas the data from the second phase of the study were qualitative analyzed using the analytic induction method. Following the principles of the analytic induction, the process of data analysis aims at identifying the main topics describing the query behavior and then compare them with theoretical structures. Practically, during interviews researchers were checking which of the sub-processes appeared as students solve the task and furthermore they were taking notes on the way that participants' actions were related to the sub-processes.

RESULTS

The results of the present study are organized in two parts, following the stages of the methodology. Firstly we aimed to categorize the sample of the study in levels of mathematical creativity. Quantitative analysis provided four groups of students (Group 1: 21, Group 2: 51, Group 3: 74, Group 4: 36 students).

Look at the follow number machine. You have to find as many pairs of numbers you can, that may convert in the same way as 2 turned out to be 8.



CAUTION: The number 2 may turn out to be 8 in different ways.

Figure 1: The task used for the interviews.

By comparing groups of students' creative abilities (fluency, flexibility, originality), there was a noteworthy difference ($p<.05$) on the number of the proposed solution between the three groups of students (Groups 1 and 2 provided similar number of solutions). Moreover, there was a difference on the number of different mathematical ideas that were employed in the solutions. Differences appeared between Groups 1 and 2 and also between Groups 2 and 3. Furthermore, more original solutions were proposed by Group 4. Secondly, we purported to investigate the process of creative thinking in mathematics. The analysis of the interviews aimed to identify the sub-process recommended by Sheffield (2009), and furthermore to compare these sub-process among students with different degree of mathematical creativity. Therefore the results have been organized according to the five sub-processes.

Investigation

The sub-process of investigation was absent in students of Group 1. Students just thought which operations can be directly performed in order to get 8 from 2.

S386: I can use other operations apart from multiplication…I can use addition and subtraction… 2+6=8 … But how can I subtract a number from 2 and get 8? I don't know!

Students of Group 2 flexibly used the four operations and their combination as well.

S79: I will use the four operations, in order to find better solutions.

S63: I am thinking to use different combinations… I have already used multiplication in combination with addition, now I am thinking to use multiplication with subtraction…

The process of investigation was similar in Groups 3 and 4. Students' investigation focused on the number of symbols and in the types of numbers that can be used in their solutions.

S283: Apart from using two operations I could use three, four, five symbols together in order to conclude with an interesting solution.

S44: I just broke my answers into three categories: integers, decimals, and fractions and each time I used numbers from different category or by mixing them.

Creation

The solutions proposed by Group 1 were limited in the ideas of 2X4=8, 2+6=8, in repeated additions or multiplications.

R: Can you think other solutions beside 2X4, 2+6?

S282: There are no other solutions…

The sub-process of creation was not differentiated between students of Groups 1 and 2 with regard to the quantity of solutions. Instead differentiation was occur in the number of mathematical ideas that was exploited in students' solutions. Group 2 proposed solution of the form 2X4 and 2+6 or solutions resulting from the analysis of the previous ones.

Students from Group 3 suggested more solutions than their peers belonging in the two previous groups, using more mathematical ideas. These students analyzed the operations of 2X4 and 2+6, by using more complex procedures and by using different types of numbers (e.g. decimals, fractions). Group 4 proposed even more complex and creative solutions. It is worth mentioning that a lot of students came up with several solutions at the first glance and without special attempt. Indicative solutions are presented below.

S122: 2+(10X1):2+2-1=8, 2+[(10X5)-40]:2+2-1=8

Relation

The sub-process of relation was not evident in Group 1. In every question of the interviewer on identifying relationships between the solutions, students responded negatively or proposed misplaced connections. The sub-process of relation was also absent from the process of creative thinking of students belonging in Group 2. Although students theoretically referred that previous solutions or preexisting knowledge helped them to identify solutions they failed to confirm their claims in actions.

S80: 2 + 4 +2=8 ... 2 +5 +1=8

R: How do you found these answers? Do they have something in common?

S80: There is no relation between them.

The majority of students in Group 3 reported relations during the production of ideas. Students found relations between their solutions with similar structure.

S57: 2 +2 +2 +2 is the same as the previous one (2X4).

S285: 2x1+6=8, 2x2 +4 = 8, 2X6-4=8, 2X5-2=8, 2X7-6=8…

R: How did these solutions come to your mind?

S285: It is like the previous one. It is the same process but with different numbers.

All students of Group 4 showed evidences of relation in their work. Relation was evident in two axes: mathematical ideas – actions and experiences, as presented below.

S88: To subtract and then add ... 2-2 +8=8

R: Does it look like any other answer that you have already suggested?

S88: Yes, with 2+7-1= because in both cases I added more than the number I would like to get and after I subtracted ... I just changed the numbers and reversed the operations.

S131: I consider things I learned through my life. Let's say the idea of 2+6 came from the first grade of elementary school, and then I thought more complex ideas from other grades.

R: So you think, I did this in first grade that in second grade...

S131: No I think clever ways that passed through my life. Maybe something my mom said, something I learned through tests…

Evaluation

During evaluating solutions, students of Group 1 acknowledged that their ideas had not something creative; however they couldn't propose a better one. Regarding Groups 2 and 3, there was not any differentiation in the process of evaluation. During evaluating their responses, students took into account the characteristics of operations and numbers used.

S12: I think this is the best solution because it has four different numbers and four different operations ... anyway this solution it will be difficult for someone to find the result.

S381: I think 2x20 – 32=8 is the smartest solution because it has the biggest numbers.

S34: This solution is the best because I combined decimals and integers.

The evaluation criteria of Group 4 did not differentiated from the previous group. However, the evaluation stage worked as an incentive for students to continue finding solutions.

S41: When you told me to find something that will not be proposed by other students, I thought that many students fear the division so they will not use it in their solutions...

Communication

The stage of communication did not appear in Groups 1 and 2. Specifically, students used merely vague comments without giving specific details on how they thought or worked.

S386: Every time I used different ways.

S14: I did 2X4=8, 2+6=8.

Students of Group 3 referred to the basic mathematical ideas they used in their solutions, sometimes by giving enough and sometimes less details.

S291: At first I thought 2+6, a simple solution. Then I saw that I could use 2X4 also. I started thinking multiples and dividers, e.g. 2X4 = 80:10 = 8. After I said that 10 is a key element that must be presented in my next solutions. Then I remembered a previous mistake and I got an inspiration…

During communication, students of Group 4 had a clear picture of their work and thus they were able to give many details.

S12: I thought to use as many symbols as I could… to do first multiplication or addition that will increase the initial number and then to subtract a number to find the answer.

S25: In the beginning I proposed the simplest solution: 2X4. Then I thought to differentiate my solutions by breaking 4 in other numbers (X8: 2) then I continued by increasing the numbers and operations...

DISCUSSION

The aim of the present study was to investigate the process that took place as individuals worked on a mathematical creativity task. For identifying the sub-processes that surface during mathematical creativity task, Sheffield's theoretical model (2009) was adopted. In order to examine the differences that exist in the creative process, we compared the sub-processes among students with different level of mathematical creativity.

Investigation refers to an in-depth study of the available information. Students with low mathematical creativity did not show any evidence of this sub-process. On the contrary, the investigation was evident in the other three groups of students. This result may be due to the fact that the more creative a student is the stronger his/her mathematical content background is. By comparing the creative ability of the four groups of students, one can observe that as flexibility was differentiated, the extent of appearance of the sub-process of investigation was also varied. According to Vidal (2009), flexible thinking requires the ability to switch between ideas and to exploit different approaches and perspectives of a problem.

At the stage of *relation* processes such as comparing ideas and combining available information evolved. Students with low and moderate mathematical creativity failed to relate mathematical ideas or made erroneous connections. Students with above average mathematical creativity managed to relate mathematical concepts, operations and numbers, to connect actions and identify similarities and differences in their solutions. Additionally, as the number and type of relations increased the number of the proposed solutions (fluency) also increased. This is probably due to the fact that the stage of relation helped students to combine ideas, compare and modify solutions, leading to numerous responses (Vidal, 2009).

The stage of *evaluation* engaged students in a process of reflection. Students with low and moderate mathematical creativity did not feel confident to justify the quality of their solutions. The most creative students were able to reflect on what they had done and furthermore the evaluation stage encouraged them to keep trying in order to find more original solutions. Indeed, Lubart (1994) concluded that students who evaluated their work exhibited higher creativity than their peers who felt that evaluation was not necessary.

The stage of *communication* involves metacognitive skills, since students were asked to reflect on their thinking. For students to be able to explain their thinking, they should have understood the procedure followed. The more able students, who had applied the stages of relation and investigation, appeared to have a more straightforward way of working and thus during communication they were detailed. Students belonging in the other two groups provided general comments or simply mentioned the solutions they had written. This may be due to the fact that they had no thinking and working plan.

The stage of *creation* was connected with the other sub-process. Group 1 proposed limited number of solutions and used obvious mathematical ideas. Group 2 did no vary substantially from Group 1 in the number of the proposed mathematical solutions, but only as to the type of mathematical ideas they used. Regarding Group 3, quantitatively and qualitatively better answers emerged. Group 4 managed to handle flexible numbers and operations, to combine mathematical ideas while utilizing different parameters of the task.

To sum up the five sub-processes may describe students' creative process on a mathematical creativity task. The creative process resembles a continuum, where at one end intense creative action is taking place and on the other end no creative action is taking place. With this perspective, the creative process can take intermediate values between the two extremes of the continuum (Lubart, 2001). The differentiation of the creative process is influenced by the sub-processes that exist and the degree of processing they receive.

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THE PERCEPTION OF THE CONCEPT OF A "CHALLENGING TASK" BY MATHEMATICALLY PROMISING ELEMENTARY SCHOOL STUDENTS

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Abstract. *In the past few decades, there has been an increase in the awareness to the special needs of gifted students and specifically, the needs of mathematically promising students. One of the mathematically promising students' needs is to work with challenging tasks. One of the ways to answer those needs is encouraging the teachers to use challenging tasks. The purpose of this paper is to examine how the concept of a "challenging task" is perceived by mathematically promising students. Fifty fifth and sixth graders from the South of Israel with high achievements in mathematics participated in the study. They were asked to freely choose two tasks they thought to be challenging and one that was not, on their part. The results indicated that mathematically promising students characterized challenging tasks as difficult, unfamiliar and require non-standard thought and an abundance of time to solve. The non-challenging task was characterized as "meaningless" and required familiarity of an algorithm, memorization and repetition and was not connected to other fields.*

Key words: Mathematically promising students, challenging tasks

INTRODUCTION

Challenging tasks have an important role in devolving mathematical thinking and coping with challenging tasks helps the students develop their own abilities (Sheffield, 1999). Diezmann and Watters (2002) mentioned that for realizing mathematically promising students' potential, they should engage in challenging tasks. Such activity facilitates the development of cogitation, encourages the use and development of metacognition skills and enhances motivation. Mathematics educators and numerous scholars in the field are developing and incorporating unique programs for mathematically promising students. Applebaum and Leikin (2007) studied the perception of "challenging tasks" in math teachers and the way teachers characterize them. Later, Applebaum (2014) examined students' perception of challenge and difficulty in different stages of involvement when solving a problem.

This study presents mathematically promising students' perception of "challenging tasks". According to various challenging tasks chosen freely by the students, they characterized the tasks as challenging and non-challenging. The research was executed in two stages: first, the participants were asked to choose two tasks which were challenging in their eyes (one from a textbook and one from a different source) and one task which was not challenging. The students were then asked to explain their choices. Second, the students were interviewed using a semi-structured questionnaire.

THEORETICAL BACKGROUND

Mathematically promising students are a combination of gifted students – who have extraordinary potential and abilities – and exceling students – who reached outstanding achievements (The steering committee for the advancement of education for the gifted report, 2004, Sheffield, 1999). According to the department of gifted and gifted students in the Ministry of Education (2008), outstanding children are defined as "thinking outside the box", original and able to expose the complexity of problems and enjoy challenging and innovative problems. Outstanding students gradually develop their qualifications and ability to understand and preform at a high level. Under conditions of uncertainty, they are willing to take risks, learn from previous experience and adjust their methods accordingly. These techniques contribute to the development of the personal potential and excellence of outstanding students, which, in turn, will have a positive effect on the environment (Fisher, 2007) as challenges are important in realizing the potential of outstanding students (Leikin, 2009).

"Challenging tasks" are non-standard problems which help construct mathematical knowledge by overcoming mathematical difficulties, investing time and effort in finding the correct mathematical tools, applying them, and evaluating their own use of these tools (Brousseau, 1997). Challenging tasks allow the examination of previously- learned knowledge application which is not just a reproducing of an algorithm or a procedure that was practiced in the classroom (Hakim, Gazit, 2009). Applebaum (2014) found that non-standard mathematical problems, which have puzzle-like qualities, are more challenging to the students and may increase their interest in mathematics. Coney (2001) revealed that high order thinking cannot be developed solely with ordinary and non-challenging tasks.

RESEARCH PURPOSE AND METHODOLOGY

In the literature there are a number of researches related to the importance of using challenging tasks in teaching mathematically promising students. Some scholars also discussed teachers' views on the concept of "challenging task". In this paper we examined how the concept of a "challenging task" is perceived by mathematically promising students.

Fifty students (25 boys and 25 girls) from five elementary schools in the South of Israel participated in the study. All of them had high achievements in mathematics (grades: 90+/100). 17 were fifth graders and 33 were sixth graders. 29 out of the fifty participated in various extra curriculum mathematical activities.

In the first stage of the study, the students received homework: to choose two tasks which they considered to be challenging – one from their textbook and another from a different available source. In addition, the students were asked to choose one task which they saw as unchallenging. They were also asked to explain their choices.

In the second stage, 20 of the students were interviewed using a semi-structured questionnaire.

All of the assignments were sorted according to their content and the students' explanations.

FINDINGS

The purpose of this paper was to examine how the concept of a "challenging task" is perceived by mathematically promising students. We found that mathematically promising students characterized "challenging tasks" as difficult, unfamiliar and require non-standard thinking and a long period of time for solving (see the students' explanations in Table 1).

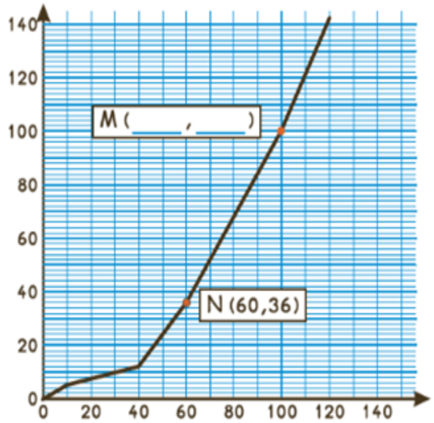
Table 1: Examples of student's answers to the question "what is a challenging task?"

A task which takes a long time to solve	**A difficult task**	**An interesting task**	**Different from other task**
St30: *[...] I needed two days to solve the riddle and finally I did it, the difficulty challenges me* St34: *[...] the assignments seemed not logical to me and it takes time* St35: *[...] it takes a long time to solve it and understand it* St38: *[...] I tried many times and only recently I succeeded* St46: *[...] because two clocks don't show the exact time, you need to think a lot and do many things and it takes time*	St41: *[...] a chose a puzzle that I couldn't solve, and that's why it's challenging to me.* St41: *[...] I chose this task because I wasn't able to do it.* St2: *[...] especially in tasks that I tried but couldn't solve.* St17: *[...] it was hard. I didn't understand the question and usually I do understand.* St24: *[...] it's hard to solve this problem and I like challenges. Not everyone can do it but I can.* St38: *[...] I tried many times and only recently I succeeded.*	St6: *[...] a task is interesting because it's complicated and takes patience.* St43: *[...] a hard and interesting task.* St44: *[...] Challenging is interesting, you gotta think and it's hard.*	St24: *[...] you need to think different, not like in a regular class.* St42: *[...] an interesting task and needs different thinking, a different style of thinking is a lot of fun.* St14: *[...] I never learned this subject so it was challenging to me.* St39: *[...] I want to learn what I don't know and that challenges me.* St44: *[...] I chose this task because it's hard and we didn't learn it in class yet.* St33: *[...] this task has percentage and we didn't learn that yet. That challenges me.*

Inquiry based problems and integrative tasks were perceived as the most challenging in the curriculum (as seen in the free choices from the textbook. See Table 2).

Table 2 : Examples of students' choices of challenging task from textbooks

<table>
<tr>
<td>The following graph describes the braking distance of a car in relation to its speed.


</td>
<td>A) Add title to both axes and the measures of point M.

B) What do the measures of point N describe?

C) What is the braking distance of a car which is going 80 kilometers per hour?

D) The braking distance of a car was 90 meters. What was its approximate speed?</td>
</tr>
<tr>
<td>The following table shows the first four elements of the series B.

<table>
<tr><td>Position in series B</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td></tr>
<tr><td>Drawing</td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>Number of rows of cubes in the element</td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>Number of columns of cubes in the element</td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>Number of cubes in the element</td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
</table></td>
<td>A) Examine the drawings and draw the fifth element of series B.

B) How many cubes are in the element you drew?

C) Fill in the last three rows of the table.

D) Find the connection between the overall numbers of cubes in each element and the number of rows and columns of cubes in each element.

E) Without drawing, predict the number of rows and columns of cubes in the sixth element. Rows: ___________ Columns: ___________

F) Check your prediction by drawing the element.

G) Think: how many cubes are in the tenth element of series B?</td>
</tr>
<tr>
<td>The famous mathematician Joseph-Louis Lagrange once stated that every natural number (a whole number, not a fraction) can be represented by the sum of one, two, three or four squares of natural numbers.

Example:

7=4+1+1+1

$7=2^2+1^2+1^2+1^2$</td>
<td>Write the sum of these numbers as a sum of one, two, three or four squares of natural numbers.

A) 26=

B) 12=4+4+4

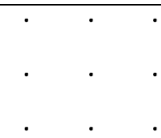
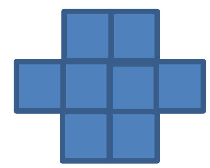
C) 32=

D) 20=

E) 35=</td>
</tr>
</table>

Puzzles were the most common example of challenging tasks from other sources (see Table 3).

Table 3: Examples of challenging tasks from non-textbook sources.

Join all of the dots using four lines and without lifting the pen from the paper.	
Fill in the numbers 1-9 in a way that no consecutive numbers touch each other.	
A spider is climbing a 190 meter smooth wall. Every day it climbs three meters up and every night in slides two meters down. How long will it take the spider to climb the wall?	

In contrast, the students characterized a "non-challenging tasks" as "meaningless", only requiring knowledge of an algorithm or repetition and memorization and unconnected to other fields of mathematical study (See Table 4).

Table 4: Examples of non-challenging tasks.

Draw a line from every fracture to the appropriate percentage:	1 $\frac{1}{4}$ $\frac{3}{5}$ $\frac{3}{4}$ $\frac{1}{2}$ 25% 50% 60% 75% 100%
Write these numbers in words:	12,784 4,085 17,802

These findings correlate with findings by Coney (2001) and Applebaum (2014), which reveal that interest and high order thinking cannot be developed solely with ordinary tasks and thus it is important to integrate challenging tasks in mathematics classes.

CONTRIBUTION OF THE STUDY

In light of these findings and in addition to them, we recommend that the authors of textbooks and developers of mathematics curriculums integrate, as possible, challenging tasks having characteristics indicated by mathematically promising students including those from non-textbook sources (puzzles and riddles, for example).

Future research will examine a larger sample and present questions of gender difference in characterization of “challenging tasks”; the effects of age on the perception of “challenging tasks”; and the effects of extra-curriculum-school activities on the perception of “challenging tasks”.

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USING OPEN-ENDED PROBLEMS AND PROBLEM POSING ACTIVITIES IN ELEMENTARY MATHEMATICS CLASSROOM

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Abstract. *The paper discusses some possibilities of using open-ended problems and problem posing activities in primary mathematics teaching as tools for fostering creativity and enhancing mathematical thinking. In the first part of the paper we will give a brief overview of the role and importance of open-ended problems and problem posing, and we will point out some of their advantages and disadvantages. Finally, we will suggest some concrete examples and ways of including these problems and activities into elementary mathematics classroom.*

Key words: open-ended problems, problem posing, mathematical creativity, primary school

INTRODUCTION

Mathematical creativity is the topic that is often neglected and found impossible to achieve in mathematics classroom. On the other hand, fostering students' creativity in mathematics education has always been a hot topic for mathematics educators and researchers (Li & Li, 2009). Is creativity something that we are born with or something that can be taught or learned? William Ward claimed that "creativity is not a mysterious, unobservable process, nor is it an innate, unlearnable ability" (as cited in Mihajlović & Vulovic, 2010, p. 131). That is why it became important to find ways to develop and stimulate creativity, especially in modern society where creativity represents one of imperatives. We can influence some components of mathematical creativity by solving certain and adequate mathematical tasks. In majority of curricula, all over the world, mathematical problem solving represents crucial part of mathematics education. What do we consider as a problem? A problem occurs when a student is confronted with a task, usually given by the teacher, and there is no pre-determined way to solve the problem. Kantowski (1980) says that a task is a problem if its solution requires that an individual combines previously known data in a way that is new at least for the solver (Näveri et al., 2011). However, the practice of solving problems in school mathematics promotes in students a suspension of sense making (Bonotto, 2013). According to Bonotto, we should rethink the type of problem-solving experience we present to our students if we want to help them to face real-life situations and challenges of the modern world.

In this paper we will discuss one method of fostering mathematical creativity and enhancing mathematical thinking. The main goal of the paper is to point out the importance of using open-ended problems and problem posing activities in elementary mathematics classroom. Since there are not many such examples in mathematical textbooks, and teachers usually state that one of the reasons for not using these kind of activities in their teaching is the lack of adequate reference material, we will give some concrete examples with brief reflection on their roles and benefits for the development of students' creativity and mathematical thinking.

OPEN-ENDED PROBLEMS AND PROBLEM POSING IN ELEMENTARY MATHEMATICS CLASSROOM

The open-ended approach was first developed in the 1970's in Japan. According to Hashimoto (1997) it provides students with "experience in finding something new in the process" (p. 86). Problems, used in *the open-ended approach, are selected so that they exemplify a variety of methods in getting answers or solutions. The aim of open-ended approach is to foster both mathematical thinking of students in problem solving and their creative abilities to the fullest extent. Each student must be allowed freedom to progress in problem solving activities according to their individual abilities and interests. Problems, used in open ended teaching, are giving opportunity to both high ability students to take part in a diversity of activities and to students with lower abilities to participate and progress according to their own abilities. Students also can use different strategies, and chose those they feel more confident with.* Majority of the questions traditionally used in mathematics teaching request from students to give an answer in a form of a single number, figure, or mathematical object. Since the expected answers are predetermined and specific, these kinds of questions are called closed ended. They are also called well-defined problems because their answers are either correct or incorrect and the correct answer is unique (i.e. What is the *perimeter of* a *rectangle* having side-lengths of 5 cm and 3 cm?). *Problems used in open-ended teaching are open-ended problems.* These problems (also called incomplete problems or ill-structured) allow a diversity of correct responses and incite a different kind of student thinking. Open-ended problems can be multiple solution tasks or they can have several ways to find correct answer (s). (i.e. *Draw* different *rectangles* with a *perimeter* of 16 cm.). Pehkonen (1999) states that "tasks are said to be open if their starting or goal situation is not exactly given" (p. 57). Nowadays it is accepted that open-ended problems represent a useful tool in the development of mathematics teaching in schools, in a way that emphasizes both understanding and creativity (Pehkonen, 2007). Another important aspect of these problems is that they also enable each student to work on the same problem according to his/her abilities. This can give students a feeling of success, achievement and fulfillment, even to those with less developed mathematical abilities. According to Guilford (1976), one of significant characteristics of creativity is divergent thinking, and since open ended problems stimulate diverse thoughts, they contribute towards enhancing divergent thinking (Kwon et al., 2006). While searching for a variety of solutions and different ways of solving a problem, students are putting a lot of ideas freely (fluency), they are trying to make up new strategies for solving that problem when old methods fail (flexibility), and sometimes they come up with unexpected, unusual and smart ideas (originality). And all these features are important components of creativity. Pehkonen (1997) distinguishes a few types of open ended problems: investigations (a starting point is given), problem posing, real-life situations (with roots in the everyday life), projects (larger study entities, which require independent working), problem fields (a collection of contextually connected problems), problems without a question, and problem variations ("what-if"-method). Open-ended problems are not limited to particular educational stage; they can and should be thought from elementary to university level. There is ample research pointing out the positive effect that use of open-ended problems has on creativity and mathematical thinking of students (Bonotto, 2013; Kwon et al., 2006; Lee, 2011; Mihajlović, 2012), but not much has been conducted with students of elementary grades. In this paper, however, we focus on elementary school grades. It is important that problems should be gradually given to students. The level of problem field depends on students' answers and solutions. Advantages of using open-ended problems are many: students participate more actively in lessons, they express their ideas more frequently, each student can answer the problem in his/her own and unique way, students are developing mathematical and creative thinking. Moreover, those problems give students more opportunities to make comprehensive use of their mathematical knowledge, skills and abilities (Burghes & Robinson, 2009). One more benefit is also that the introduction of this type of problems to the classroom may bring mathematical education one step closer to real-life mathematics. Bonotto (2013, p. 53) says "that less structured, more open-ended tasks could foster flexible thinking, enhance students' problem solving skills and prepare students to cope with natural situations they will have to face out of school". Still there are some disadvantages of using open-ended problems. According to Klavir and Hershkovitz (2008), teachers do not usually possess either the tools to evaluate the work of the different students or the tools for promoting higher levels of problem solving. There are also some other disadvantages such as difficulty of successful problem posing, difficulty of developing meaningful problem situations, and difficulty of summarizing the lesson. Majority of elementary teachers in Serbia (teaching in grades one to four of elementary school) are not trained enough to use and create open-ended problems (Dejić & Mihajlović, 2014). Some studies point out that a very low percentage of elementary teachers in Serbian primary schools is familiar with the use of open-ended problems and the problem-posing approach as tools to stimulate creativity (Mihajlović, 2012). Mihajlović conducted a study among 157 primary teachers in Serbia as a part of a larger research. One of the questions the author dealt with was to determine what ways primary teachers use to stimulate mathematical creativity in students. She found that about 14% of teachers use situations when they ask students to make up the story problem for a given expression or equation. The author also found that although majority of teachers (94.82%) had replied that they used open-ended problems more or less in their teaching work, only 14% of teachers had given a correct example of such open-ended problem. The largest percentage of teachers had either given an example of closed problem instead or they had not given any example at all.

SOME EXAMPLES OF OPEN-ENDED PROBLEMS AND PROBLEM POSING IN ELEMENTARY MATHEMATICS CLASSROOM

Primary teachers face difficulties in creating relevant and purposeful open-ended problems which have value in a mathematical sense, and may foster creativity and may provoke higher-order thinking in students. In this part of the paper we will give some examples of using open-ended problems and problem posing activities in primary grades which can encourage students' creative thinking and enhance mathematical thinking.

Example 1. *Consider the following sequence: 1 2 4 6 8 12. Find a number that is different from others. If possible, try to find as many as possible answers.* There are few possible solutions. Children might choose as an answer 12, because it is the only two-digit given number. Or they might decide to choose 1, because it is the only odd number in the given sequence. However, answer might be 8 as well because all other numbers are factors of 12, etc. This problem for example gives students an opportunity to develop their fluency (number of answers), as well as originality (rare and useful answers).

Example 2. *Matija has built a tower with playing cubes. The front view is shown in the first picture and the side view is shown in the second picture (Figure 1). Draw how the tower might look like. Try to give more than one answer.* This problem investigates the relationship between 2D and 3D shapes and gives students opportunity to search for different solutions. It also helps students to visualize 3D shapes from 2D representations and shows that different 3D objects can be represented by the same set of 2D objects. Teacher can additionally ask students to create their own tasks which are in some way similar to the original one. Such extension of the problem may also foster some creative thinking components (i.e. fluency and flexibility).

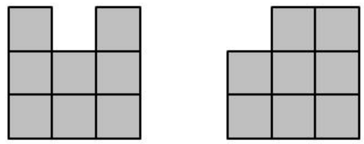

Figure 1: View from the front and from the side

Example 3. *Imagine a two-digit number. Swap the places of the digits of this number and find the sum of these two numbers. Find as many two-digit numbers as you can which when summed with their reverse number give the two-digit sum (i.e. 35 + 53 = 88). Can you notice some pattern? Extension of the problem might be to ask students to find the patterns if the result is three-digit number.* We can let students first work individually on the problem and after that to share and discuss their findings in the group. There is no one single solution to this problem, nor one determined strategy or method, so the students must find their own way to approach the problem. In solving the problem students may discover different patterns, such as: (1) They may notice that in order to have two-digit result, the sum of the digits of the number must be less than 10; (2) The value of the digits of the result depends on the value of the sum of the digits of the two-digit number; (3) All sums represent the multiples of number 11. Etc. Through production of various solutions, ideas and approaches, students may enhance the flexibility in their thinking.

Example 4. Teachers always tell students that mathematics is used everywhere in real world, but do they create an environment that proves this point? We should encourage students to write essays or reports about using mathematics in everyday life (i.e. *Where can you see mathematics at work on your way home? How do you use mathematics outside of school? What measurement units your mom or dad use and how?* etc.). We can thus stimulate and challenge students' interest for mathematics and also, we can find out what students have already learned.

Example 5. *Nikola and Sasha decided to earn some extra pocket money by helping their uncle in the shop over the holidays. The uncle told them that he would pay them in the following way: Sasha would be paid 10 euro for the first working day, and each next day she would get 2 euro more than the previous day. For the first day at work Nikola would get 1 euro, but each next day his salary would be doubled compared to the previous day. Whose place would you rather be, Sasha's or Nikola's, and why? After how many days will their uncle run out of money?* This problem does not have only one correct and definite solution. In fact, it is not specified how many days children should help their uncle. If they work 6 days or less, Sasha will earn more money. If they work more than 6 days, Nikola will earn more.
The goal of solving problems of this type is that children compare the growth rate of two different sums. This problem also has real life context, because sometimes we have to make everyday decisions about what is more profitable. The other part of the question is also open-ended, and there is no precise answer. Children should realize that adding up the double

numbers makes the total sum grows faster. One of the possible ways to track results and to compare the growth is to use a table. This way, students can calculate and write the amount of earned money for the first few days and then make conclusions. If we denote number of days the children help their uncle with n, then on the n^{th} day Sasha will get $10 + (n-1) \cdot 2$ euro, and Nikola will get 2^{n-1} euro. When n tends to infinity the function $f(n) = 2^{n-1}$ grows faster than the function $f(n) = 10 + (n-1) \cdot 2$, and so the value of the sum $\sum_{n=1}^{\infty} 2^{n-1}$ compared to the value of the sum $\sum_{n=1}^{\infty}[10 + (n-1) \cdot 2]$. Of course, the main idea is that children realize this by presenting and solving the problem in their own way. Since, after 20 days the total sum will exceed one million euro, Sasha and Nikola's uncle will most likely be bankrupt.

Example 6. *We tell students that we will play a two-player strategic game. First, we explain them the game rules: At the beginning of the game we put 5 counters in a row. Then, players take turns taking counters. Each time, the player has the right to take one or two counters (optional). The winner is the one who takes the last counter. Pupils' task is to answer the following questions: 1. Can you find out how the first player should play in order to win the game (a winning strategy)? 2. Is the game "fair"? (The game is "fair" if each player has an equal chance of winning.) 3. By changing some rules or parts of the game, create your own game.* The last requirement actually presents the problem posing activity. In order to help students we can use the strategy proposed by Brown and Walter (1983), "What-if-not?". Teacher can ask some additional questions such as: What would happen if the number of counters at the beginning of the game was even? What would happen if players took one or three counters in each turn? Elementary school students will mostly come up with the answers by experimenting. In higher grades of primary school teachers can present Nim game to the students and this can be a starting point for open-ended activities.

CONCLUSION

Most mathematical questions and tasks in mathematics have only one answer. That is one of the reasons why they discourage students from using and investigating various ideas. On the other hand, open-ended problems have advantages as they allow diverse answers or diverse ways of solving problems. One more advantage of those problems is that every student, no matter if he/she is highly capable or struggles with mathematics, can try and find his own solutions to the problems depending on his own scope and level of abilities. Therefore open-ended problems are particularly appropriate for classes of differentiated teaching. They represent powerful means for encouraging and fostering creative mathematical thinking and they should be implemented in school starting with early age. Although not enough attention has been paid so far to the development of mathematical creativity in our country, we believe that this can be improved. Besides organizing seminars, attention should be paid to introducing appropriate contents, open-ended problems, and problem-posing activities in mathematical textbooks. All of these imply that special attention should be given to primary-teacher training at the university level, because the teacher is, and remains, the major factor of students' achievements in the educational process. Furthermore, we hope that more research on this topic will be conducted with students of elementary grades.

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ORAL PRESENTATIONS 1.2.

Creativity in Mathematics Competitions

Chair of the session: **Roza Leikin**

THE ANALYSIS OF 6TH GRADE STUDENTS' WORKS AT THE OPEN MATHEMATICAL OLYMPIAD

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Abstract. *The paper presents the results of research on the 6th grade students' works at the 39th Open Mathematical Olympiad. These results are the first stage of the currently ongoing study on the Olympiad performance of those students who have participated repeatedly in grades 6, 8 and 9. The purpose of the study is to evaluate the student's progress in problem solving. This article demonstrates a problem solution classification scheme which will be applied in evaluating the mathematical and general thinking skills of an individual student. The section on results provides consolidated statistics on each classification criterion.*

Key words: Mathematical Olympiads; problem solution; students' performance; thinking skills.

INTRODUCTION

Latvia is a small Baltic state which is presently striving to achieve economic prosperity. To ensure this, highly qualified scientists, economics and engineering specialists are required. However, one of the most crucial problems is the decrease in the general level of school student's mathematical knowledge, which is worrisome to university and technical college professors. Latvian students demonstrate quite an average level of proficiency in mathematics in international comparison as well. For example, PISA 2012 study shows that only 8% of Latvian participants are top performers, and the mean score of our participants is below the average OECD score (OECD, 2014).

The quality of mathematics education can be improved in two meaningful ways: on the compulsory level, by enhancement of mathematics curriculum, and in extracurricular activities where the students can master their skills to reach higher results in Olympiads. According to the newest mathematics curriculum, every student should acquire the basic concepts of problem solution. In their turn, the students' Olympiad works demonstrate what basic skills they have learned, what methods they use, and what are their beliefs on problem solutions.

The students who actively participate in Olympiads may be called mathematically interested students; potentially, they are among the top performers. The purpose of the study undertaken by the authors of this paper is to establish the progress of individual students in solving problem tasks, to find out whether the students have learnt new solving methods, and whether original ideas can be found in their works.

The Open Mathematical Olympiad (OMO) is one of the most popular educational Olympiads in Latvia. It has been organized by the Extramural School of Mathematics at the University of Latvia starting from 1974. The Olympiad, where any student can prove their abilities, takes place on the last Sunday of April. Every year around 3000 students take a part in this event. The assessment of students' performance in OMO is intense. The solutions have to be inspected by the Olympiad jury during two weeks. Evaluated works are returned to the students at the awards ceremony. The short time when these materials are accessible is one of the reasons why a complex investigation of students' works is problematic. General comments on students' achievements or failures (Bonka at all, 2010), detection of the application of general reasoning methods in problem solutions (Bonka, Andžāns 2004); investigation of problem sets (Avotiņa 2011, Šuste 2009) are discussed in a number of scientific papers. As part of this study, it was possible to photograph the works of 6th grade (OMO39) and 8th grade (OMO41). The works of 9th grade (OMO42) will be photographed in May this year.

METHOD

First statistical investigation of 6th grade students' performance on OMO39 was reported at the Baltic conference (Ramāna at all, 2014). Therein was offered a general framework that includes 3 main components of the problem solution: the stage of comprehension, the stage of elaboration, and the stage of presentation. Each of the stages was expanded for every separate problem. Thus it was possible to evaluate all 446 students' works together and to determine the average qualitative indicators. In planning further work, it was important to pay more attention to the students' mathematical knowledge and the methods used in solutions, which would characterize more exhaustively the individual participant of Olympiads. This article presents a detailed research model to classify the activities performed in solving a problem. The results section presents a summary of statistical characteristics.

Principles by Bloom's Taxonomy	Explanation	Levels of competencies
Conceptual understanding Pre-processing of data	Demonstration of knowledge and comprehension of the solver Application of facts, concepts and skills to the given data	Level 1
Analysis	Investigation of the properties of given objects and common process using different methods	Level 2
Synthesis	Posing of the hypothesis, construction of algebraic formulas, generalization	Level 3
Evaluation	Reasoning, estimation, explanation, justification	Level 4

Table 1. Description of levels of competencies used in the research of students works

To evaluate the students' skills in solving the given tasks, there were adapted the principles of revised Bloom's Taxonomy (Anderson, Krathwohl, & Bloom, 2000). The following sequence of levels related to the students' competences and thinking skills correspond quite well to the general process of problem solving (see Table 1).

B. Rott, when describing studies of the problem-solving process, has developed a broad review of literature on problem solving from both psychological and mathematical didactics perspectives (Rott, 2013). A. Schoenfeld's structured *stepping stones model* that Rott refers to, which describes solution as a path through problem space, corresponds to the expanded framework of our study. The solution presented by the student is like a mosaic composed of various operations or activities. Those can be classified into several levels of competence.

Activities of the first level demonstrate initial operations with the given values, also characterizing the solver's comprehension and knowledge. In the structure presented here (see Table 1), conceptual understanding and pre-processing of data are combined into one – the first level of competence. Other levels of competence are further named accordingly as the second, third and fourth levels. The common set of activities gives the opportunity to evaluate the solution of each individual student as a flow of activities both in depth and in breadth, by accounting for both the productive activities leading to the solution, and the unproductive ones, and to classify them according to the various levels of competence. Considering the fact that Olympiad problems are different, there was a separate set of activities prepared for each problem (see Table 2). The characteristic deficiencies and errors of solutions were coded separately.

PROBLEMS

The 6th grade problem set consists of 5 problems. The results of the first 3 are presented here. The reason for such preference is such that the students' works presented more activities in the solutions of these problems (solution of the 4th problem was mostly based on the method of trial and error, while the 5th problem was not understood by more than 80% of students).

Problem 1. There are numbers 1 2 3 4 5 6 7 8 9 10 on the black-board. It is allowed to erase any two of them (say a and b) and write the number $a + b + 2$ instead of them. The process will be repeated until there is only one number left. Prove that the same number will be left independently of the sequence of operations. Determine this number!

Problem 2. Dissect the square into two equal polygons - a) hexagons; b) heptagons.

Problem 3. What is the smallest quantity of cells (here: cell is a unit square) that must be colored green in a square consisting of 8 x 8 cells, so that in the uncolored part it would not be possible to place any rectangle consisting of 3 x 1 cells without covering any green cell?

RESULTS

Every problem solution was divided into a set of activities such as arithmetical calculations, algebraic transformations, drawings, grouping of elements, geometrical constructions, etc. Every activity is defined as a fragment of a student's work (text, calculations, formulas, pictures) with one certain meaning: for example, calculation of the sum of numbers. These activities were distributed according to the appropriate level of competency. Table 2 contains the descriptions of main activities for all three problems. There are given the percentages of students who completed each of the activities. These results show that solvers had a rather good comprehension of the given objects (see Table 2, Level 1), whereas between one-fifth or one-third of the participants did not understand the givens of the problem at all (see Table 3).

	Problem 1	Performers (%)	Problem 2	Performers (%)	Problem 3	Performers (%)
Level 1	Execution of the first steps of the procedure	61	Drawing of -polygons -hexagons -heptagons Dissection of a square	 97 90 70 80	Drawing of rectangles 1 x 3 Coloring of cells	62 90
Level 2	Common sum of the given numbers Changes of the common sum Construction of examples Detection of resulting number	53 39 67 57	Investigation of -midline -diagonal -polyominoes -symmetry Construction of polygons inside the square	 84 45 18 37 15	Calculations Decomposition of a square Additional coloring of cells Special properties of layout	30 30 46 3
Level 3	Detection of algebraic rule	34	Dissection in hexagons Dissection in heptagons	72 39	Minimal number of green cells	90
Level 4	Justification of the algebraic rule Proof of the invariance	25 7	Correct answer Only hexagons Only heptagons	36 36 3	Visual presentation -relevant coloring -correct coloring Proof that the coloring is minimal	 17 31 13

Table 2. Activities in the solutions of problems done by the performers (students) according to the four levels of competencies

Considering the Analytic level and especially the level of Evaluation in Table 2, the total number of performers decreases. Experimentation and research on the objects' properties led to the posing of some particular hypothesis. Part of the students stopped at the Analytic level. Students who found some general properties of given objects not always elaborated these findings completely. Others who detected the rules of the whole process showed good problem solving skills (see Table 2, Level 3). Unfortunately, not all of them can reach the Evaluation level because they did not present correct proofs, justifications or explanation of the acquired results.

Problem 1 seems like an easy calculation task. The challenge here is to determine the general properties of the given process to prove the invariance of the final result. The most popular method in solving this problem was the construction of the complete example (or more examples). Part of students did productive actions to conclude the regularities of the process from these examples (see Table 2, Level 2). 29% of participants had misconception about the solution, they presented an example as the complete answer (see Table 3).

Problem 2 is one of the best-solved problems in the 6^{th} grade group. The answer does not need explanation or proof – just a correct dissection of the square. 16% of all participants drew the correct answers straight away. 39% of participants found only one answer (see Table 2, Level 4). Some students demonstrated longer experiments in their drafts, sometimes constructing many polygons - even more than 24 examples. Unfortunately it did not lead them to the correct answer always.

To solve *Problem 3* it is necessary to detect and justify the minimal number of colored cells, to find the appropriate coloring, and to prove the universality of this coloring. Arithmetical calculations and/or layout of the rectangles in the square were the main methods to detect the minimal number of green cells. The misunderstanding of the layout properties characterizes the students' mental level of abstraction – 24% of students observed only concrete static models (see Table 3). 46% of students used the additional coloring in 3 colors that can help finding the correct green coloring of cells but only 9 students explained the properties of this approach.

Problem 1	Performers (%)	Problem 2	Performers (%)	Problem 3	Performers (%)
Do not understand the problem	35	Do not understand the problem	19	Do not understand the problem	39
Have arithmetical and logical mistakes	12	Dissect the square in unequal parts	14	Cannot dissect the square in the rectangles correctly	65
Consider an example as the complete answer	29	Construct figures inside the square	6	Coloring of the fixed dissection	24
		Mix up the terms	3		

Table 3. Characteristic faults in the students works

CONCLUSIONS

The presented research shows that approximately 40% of students demonstrated a high level of thinking skills. Nevertheless, most of them struggled with the correct application of mathematical language. The students had most misconceptions and gaps in the problem solutions field. The percentage of mistakes related to basic mathematical knowledge is relatively low. There were only a few original solutions: for example, to solve Problem 2, a student drew two equal polygons inside the square and then filled the empty regions until the correct composition was reached. In another example, to analyze the properties of the process described in Problem 1, the number of addends was replaced with different whole numbers.

The presented results summarize the most popular activities in the solutions of three given problems. Here offered framework will be verified and approved in the further study to better characterize the abilities of an individual student.

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LEARNING MATH THROUGH RESEARCH AND COOPERATION

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Abstract. *MATh.en.JEANS/ MeJ (Méthode d'Apprentissage des Théories mathématiques en Jumelant des Etablissements pour une Approche Nouvelle du Savoir) are workshops for students of different age groups. These workshops encourage the students to engage in and eventually learn math by discovering and researching it. The MeJ workshop develops students' creativity, initiative, critical thinking, problem solving skills, etc., and gives students the chance to exchange ideas by working in groups both within their MeJ workshop and with students from a different MeJ workshop. MeJ workshops have a long history in France, where they started in 1985, and a very brief history in Romania. In 2013, we began to plan joint MeJ workshops through a bilateral collaboration between my school and a French high-school, which is continued via an Erasmus+ project. This paper presents our project, our experience in developing MeJ workshops and the results achieved so far.*

Key words: learn, math, research, collaboration, creativity, workshops.

THE PROJECT – IN BRIEF

During the 2013 - 2014 school year, a MeJ workshop was organized in Romania through a bilateral collaboration between Lycée d'Altitude Briançon (France) and Colegiul National Emil Racovita Cluj-Napoca (Romania) with the financial support of the French Institute in Romania, the MeJ Association (http://www.mathenjeans.fr/) and the Animath Association (http://www.animath.fr/). Lycée d'Altitude had been implementing MeJ activities since 1998 and was looking for a MeJ workshop in Romania to collaborate with. Colegiul National Emil Racovita was interested in developing a math project in collaboration with a European country, for improving the students' transversal and math skills.

The aims of our project were:

- to introduce students to a different way of doing math;
- to show students, by doing math research, that math is wonderful;
- to develop students' curiosity and enjoyment of doing math through a method which gives space to autonomy and imagination;
- to introduce students to academics so they would better understand math research careers.

The main activity of the project was to set up and implement the two collaborative MeJ workshops. The workshops involved:

- two high-schools (Lycée d'Altitude Briançon and Colegiul National Emil Racovita Cluj-Napoca);
- 20 high-school students (volunteers): 10 students (aged 17 – 18) from Colegiul National Emil Racovita and 10 students (aged 16 – 18) from Lycée d'Altitude;
- 2 teachers: Mr. Hubert Proal from Lycée d'Altitude and myself;
- 3 researchers: Mrs. Adela Lupescu from the Babes-Bolyai University, Mr. Camille Petit from the Fribourg University and Mr. Yves Papegay from INRIA – Sophia Antipolis Méditerranée research centre;

- 10 research topics proposed by the French researchers;
- weekly workshop activities in both schools – in each school, the students met with the teacher to discuss the stage of their research; students in groups of two or three carried out the research during and in between the weekly workshops;
- 2 video-conferences which allowed students to share their work with their mates from the other school;
- meetings with the researcher: 12 face-to-face meetings in Cluj-Napoca and 4 video-conferences in Briançon for discussing the status of the research;
- a presentation of the research work and results in each school;
- participation in the Lyon Congress for sharing the experience and the results of the research; the students from both schools prepared and made together presentations on each research topic;
- four research papers written jointly by the students who researched the same topics;
- publication of the research papers.

The MeJ workshop – principles and practical aspects

The MeJ workshop allows students to meet researchers and experience an authentic math-research process in school, with both a theoretical and an applied dimension. Of major importance in the MeJ workshop are the research topics. They are math-research problems which are formulated differently from the math problems the students are used to. For example, two of the topics the students researched were as follows:

The vaults. We have stones of polygonal shapes (except rectangle) and we have to build a vault between two pillars. A stone is in equilibrium if the perpendicular bisectors of the contact surfaces and the vertical line passing through the center of gravity of the stone are concurrent. To figure out how to build such a vault, study its shape. (fig.1)

Sujet 3 – **Les voûtes**. Nous disposons de pierres de formes polygonales (sauf rectangle) et on doit réaliser une voute entre deux piliers. Une pierre est en équilibre si les médiatrices des surfaces de contacts et la droite verticale passant par le centre de gravité de la pierre sont concourantes.

Comment réaliser de telle voûte, étudier la forme des constructions.

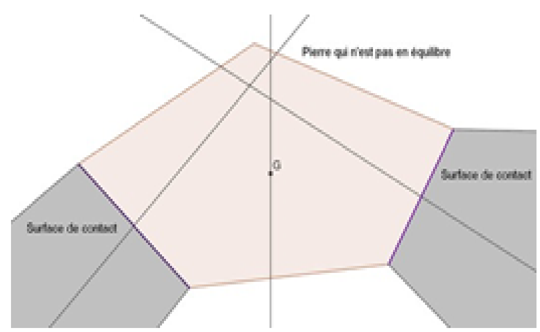


Fig. 1: Research topic 3 - The vaults

The game of Hex. Hex game is a board game played on a diamond grid with hexagonal cells. There are two players: a blue one and a red one. Players take turns placing a stone of their colour on a single cell. The goal of the game, for the 'red player', is to form a connected path of red stones linking the opposing sides of the board marked with red and vice versa for the blue player. Develop a winning strategy.

Sujet 7.1 – Le jeu de Hex.

Le jeu de Hex se joue sur un damier en forme de losange dont toutes les cases sont hexagonales. Il y a un joueur bleu et un joueur rouge. Chaque joueur, à tour de rôle, colorie une case du damier avec sa couleur. Le but du jeu, pour le joueur rouge, est d'arriver à relier les deux côtés rouges du damier par un chemin constitué de cases rouges et vice versa pour le joueur bleu.

Mettre en place une stratégie gagnante.

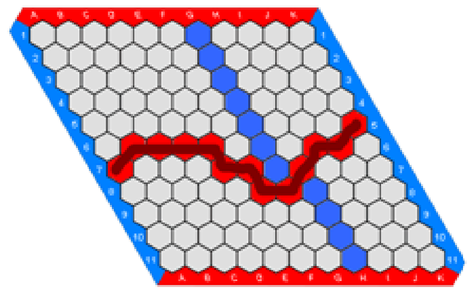


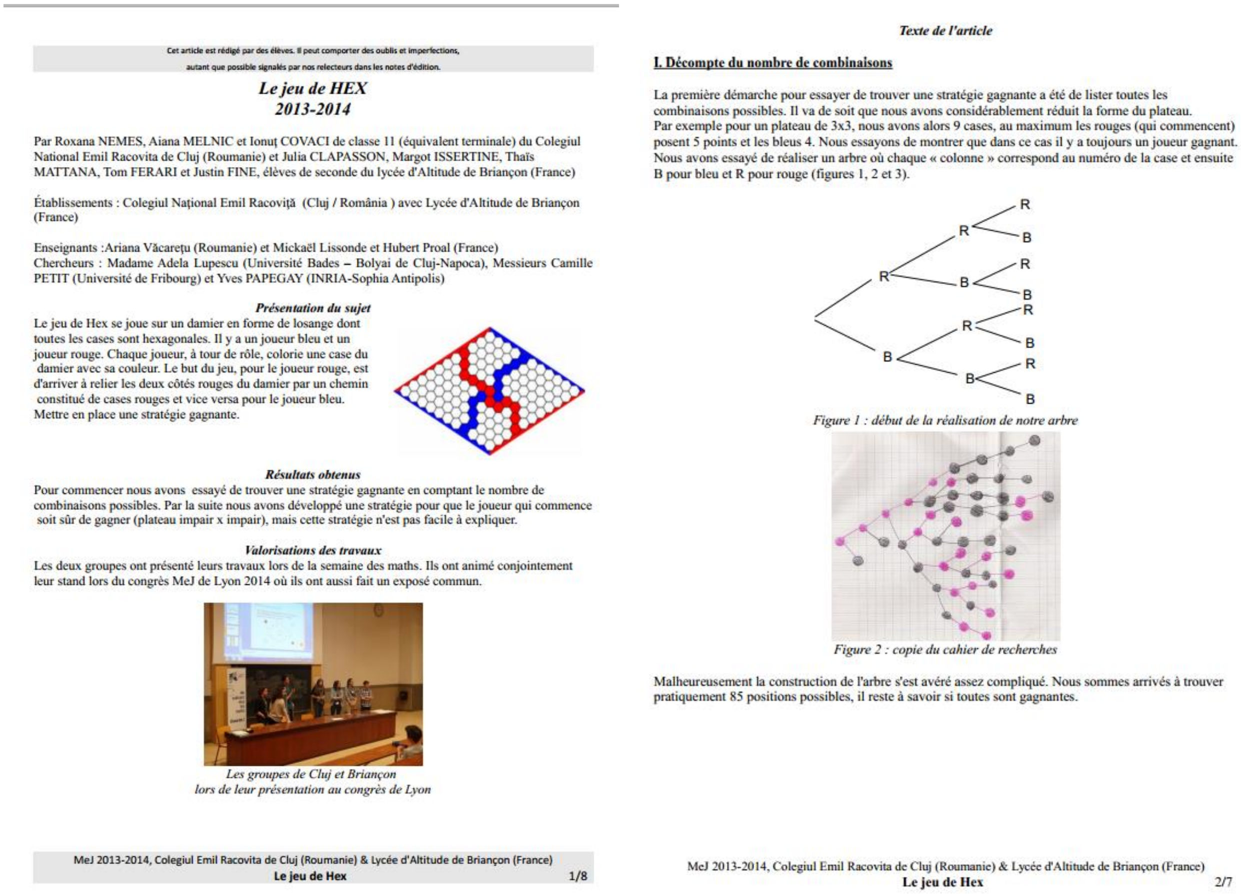
Fig. 2: Research topic 7.1 - The game of Hex

In brief, a MeJ workshop is organized as follows: it lasts throughout the school year; students register voluntarily – there is no student selection; math research topics are launched by professional researchers after being discussed with the teachers; students get a math research topic by choosing what they want to work on; students work in groups for doing the research & collaborate with students from another MeJ workshop who research the same topic; teachers and professional researchers facilitate student research; students share their research results in the annually organized MeJ Congress and different scientific events; students write and publish a scientific article about their research findings.

The MeJ Congress is an important part of the MeJ workshop. One of the reasons is very well pointed out by Mr. Hubert Proal in his report on the 2013 – 2014 MeJ workshop:

The conference is a unique moment of synthesis and evaluation which puts across all the research efforts since the beginning of the year. Its preparation gives an opportunity to each student to raise awareness of the achieved results (by himself, by his group or the twin group) and to reorganise the new knowledge related to both the content and the process of the research. To present the research work and results, students need to focus on the main results, order the stages of the research process, and find an appropriate way to reach the audience. Moreover, the students have to be able to receive the critical feedback of a wider scientific community.

A MeJ workshop is a replica of math research done by professional researchers. This is why the last stage of the MeJ workshop also includes a learning activity in which writing plays a vital role. The students write their articles, which are sent to be validated by the researcher – this is when the researcher gets a comprehensive image of the students' work – and then submitted to an editorial board. The students' articles are published in the MeJ publication or in the Quadrature magazine. You can find last year's articles at http://www.lyc-altitude.ac-aix-marseille.fr/spip/spip.php?article91.



Cet article est rédigé par des élèves. Il peut comporter des oublis et imperfections, autant que possible signalés par nos relecteurs dans les notes d'édition.

Le jeu de HEX
2013-2014

Par Roxana NEMES, Aiana MELNIC et Ionuț COVACI de classe 11 (équivalent terminale) du Colegiul National Emil Racovita de Cluj (Roumanie) et Julia CLAPASSON, Margot ISSERTINE, Thaïs MATTANA, Tom FERARI et Justin FINE, élèves de seconde du lycée d'Altitude de Briançon (France)

Établissements : Colegiul Național Emil Racoviță (Cluj / România) avec Lycée d'Altitude de Briançon (France)

Enseignants :Ariana Văcarețu (Roumanie) et Mickaël Lissonde et Hubert Proal (France)
Chercheurs : Madame Adela Lupescu (Université Bades – Bolyai de Cluj-Napoca), Messieurs Camille PETIT (Université de Fribourg) et Yves PAPEGAY (INRIA-Sophia Antipolis)

Présentation du sujet

Le jeu de Hex se joue sur un damier en forme de losange dont toutes les cases sont hexagonales. Il y a un joueur bleu et un joueur rouge. Chaque joueur, à tour de rôle, colorie une case du damier avec sa couleur. Le but du jeu, pour le joueur rouge, est d'arriver à relier les deux côtés rouges du damier par un chemin constitué de cases rouges et vice versa pour le joueur bleu. Mettre en place une stratégie gagnante.

Résultats obtenus

Pour commencer nous avons essayé de trouver une stratégie gagnante en comptant le nombre de combinaisons possibles. Par la suite nous avons développé une stratégie pour que le joueur qui commence soit sûr de gagner (plateau impair x impair), mais cette stratégie n'est pas facile à expliquer.

Valorisations des travaux

Les deux groupes ont présenté leurs travaux lors de la semaine des maths. Ils ont animé conjointement leur stand lors du congrès MeJ de Lyon 2014 où ils ont aussi fait un exposé commun.

Les groupes de Cluj et Briançon lors de leur présentation au congrès de Lyon

MeJ 2013-2014, Colegiul Emil Racovita de Cluj (Roumanie) & Lycée d'Altitude de Briançon (France)
Le jeu de Hex 1/8

Texte de l'article

I. Décompte du nombre de combinaisons

La première démarche pour essayer de trouver une stratégie gagnante a été de lister toutes les combinaisons possibles. Il va de soit que nous avons considérablement réduit la forme du plateau.
Par exemple pour un plateau de 3x3, nous avons alors 9 cases, au maximum les rouges (qui commencent) posent 5 points et les bleus 4. Nous essayons de montrer que dans ce cas il y a toujours un joueur gagnant. Nous avons essayé de réaliser un arbre où chaque « colonne » correspond au numéro de la case et ensuite B pour bleu et R pour rouge (figures 1, 2 et 3).

Figure 1 : début de la réalisation de notre arbre

Figure 2 : copie du cahier de recherches

Malheureusement la construction de l'arbre s'est avéré assez compliqué. Nous sommes arrivés à trouver pratiquement 85 positions possibles, il reste à savoir si toutes sont gagnantes.

MeJ 2013-2014, Colegiul Emil Racovita de Cluj (Roumanie) & Lycée d'Altitude de Briançon (France)
Le jeu de Hex 2/7

Fig. 3: The first two pages of the article "The game of Hex"

The weekly workshop activity in the school

When I launched the project, my main concern was how to organize and develop the weekly workshop in the school, as it was different from math lessons. It was crystal clear to me what the students had to do: work in groups; read the research topic; translate it into Romanian; discuss it; check if they have the same understanding; translate it into mathematical language; and divide the tasks among the group members. (fig. 4)

I learned from my French colleague that, as the teacher, I would always have to be there, that I should not translate the problem, either into Romanian or into math language, that I should not transform the research topic into a sequence of questions.

During these activities, I created an environment which was appropriate for research and discussions without giving answers to the students' questions. I encouraged the students to communicate their thoughts, to rephrase their colleagues' statements or questions, to allow enough time for their colleagues to talk. I supported the students to formulate questions, to find appropriate ways to model the problem to communicate the results. Thus, I found out that the teacher encourages, comforts, invites to discussions, invites to rigorous proof, (sometimes) suggests the tools if the students ask for it, advises in organizational matters and in the presentation of the results.

It was important for me to keep the students involved in the research, to insist on accuracy and on regular attendance of all students. Students registered in the research project voluntarily, but once they did, they had to do the work.

Fig. 4: Weekly workshop activity

Fig. 5: Researcher during a weekly activity

The researcher participated in almost all the weekly workshop activities (fig. 5). It wasn't necessary to do it, but she was very enthusiastic about the students' work and wanted to be with us in all the project activities. She also encouraged students to formulate questions related to the research topic, provided feedback on their work in progress, and validated, together with the French researchers, the findings of the students' research and their research articles.

Conclusions

During the one-year bilateral collaboration between Colegiul National Emil Racovita and Lycée d'Altitude, we identified the following issues:

- low and average achievers in math were very interested in participating in the MeJ workshop;
- students' motivation to learn math increased when researching math topics within the MeJ workshop; some students that were low achievers in math showed increased motivation for improving their math academic performance while/ after being part of the MeJ workshop;
- we could not provide clear evidence for the transversal skills and math competences the students developed in the MeJ workshop as we had no system (methods & tools) for assessing the development of the students' skills through the MeJ workshop;
- collaboration between students from two MeJ workshops supports students' research process and math learning; if the MeJ workshops run in parallel in different countries, the students' collaboration yields additional value, as they learn about a different culture and communicate on everyday topics and on math-related topics in a foreign language.

The students' MeJ workshop experience is well illustrated by one student's reflection:

When we joined the project in the 11th grade, we didn't have a single idea about what a mathematical project was or about how we should do our research. During the months in which we completed our project, we learned that sometimes the ideas that seem the greatest are the ones that fail first. We worked hard to get some relevant results before the Lyon Congress. At the conference, we were pleasantly surprised. The conference was filled with students, researchers and teachers from all over France, and we also had the opportunity to meet with Romanian researchers who were working there. At first, we had problems communicating with the students from France, but we quickly got over the language barrier. All in all, the whole project was for us a great experience, and we are proud to be among the first 10 Romanian students to have participated in this project.

In September 2014, we continued the implementation of the MeJ workshops as part of the Erasmus+ project Learning math and languages through research and cooperation – MatLan, funded with support from the European Commission. The MatLan project (2014-1-RO01-KA201-002699) builds on the experience gained in the MeJ workshops aiming to valorize students' creativity and innovation by inviting them to discover and research mathematics. Mathematics in the classroom is too abstract for the students to feel attracted to, but in the workshops they get to mathematically investigate issues that no-one has an answer to yet. MatLan adds a series of new dimensions to the MeJ experience: **the assessment dimension**: we will develop guidelines for assessing students' skills (transversal skills and mathematical competences) developed through mathematical research within the MeJ workshops; **the inter-cultural dimension**: we will encourage the inter-cultural exchanges around mathematical issues; **the multilingualism dimension**: we will create opportunities for the students' language learning through collaboration within mathematical research activities; **the formal education dimension**: we will prove and promote a possibility for including the MeJ workshop (non-formal education) into the school curriculum (formal education).

In September 2014, we had a very good start with the MeJ workshop at Colegiul National Emil Racovita as 78 high-school students (aged 15 – 18) enrolled in the MeJ workshop. The good start gives us hopes for the successful implementation of the MatLan project.

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NOVELTIES IN MATH OLYMPIADS IN LATVIA

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Abstract. *Latvia has more than 65 years old traditions of math Olympiad. Continuous changes in the educational system also had an impact on the content of Olympiads' problems. Usually Olympiads' problems in Latvia differ from the school standard tasks with the depth of the solution, the synthesis of different topics, as well as a greater role of the proof, so the students' training for the competitions should reflect these highlights. The school year 2014/15 is announced as "Invariants' method year", it means – problem sets in Latvian math Olympiads for each grade will contain at least one problem solvable by this method. The necessity of changes in Olympiad problems' content, problems of the regional round of Latvia State mathematical Olympiad in school year 2014/15 as well as students' achievements will be discussed in the paper.*

Key words: Mathematical Olympiads in Latvia, Olympiads' problems, general methods of proof, method of invariants.

SYSTEM OF MATH OLYMPIADS IN LATVIA

First State Olympiad in mathematics in Latvia has been hold in 1946. Since school year 1949/50 these Olympiads are annual, usually organized in 4 rounds: school round, regional round, State round and selection round for a place in a Latvian team for participation in the International Math Olympiad (IMO). Aims of mathematical Olympiad, defined by National Centre for Education (NCE) (National Centre for Education [NCE], 2014), are:

- activate and deepen the math education and extracurricular work;
- to uniform criteria for evaluating the effectiveness of work of teachers and students;
- to create additional incentives for learning of mathematics;
- to select candidates for membership in the International Mathematical Olympiad.

(*Translated by the author.*)

School and regional rounds formally are open for each student, however sometimes due to organizational problems the number of participants could be limited by the local organizing committee. The number of participants at the State round is limited by NCE, only the winners of the regional round can take part in it.

To give opportunity to more students to take part in state level competition since year 1974 is organized Open math Olympiad for 5th – 12th grade students. This Olympiad is very popular, ~3000 students take part in it each year (Latvia has less than 2 millions inhabitants, so this number is a significant part of all school students).

The main organizer of state-level math Olympiads in Latvia (both State Olympiad and Open Olympiad) is the Centre of advanced math education at the University of Latvia, called "Correspondence Mathematics School [CMS]" (see http://nms.lu.lv). State Olympiad is organized in collaboration with NCE while Open Olympiad is CMS initiative. A large number of volunteers is also involved into organization of these Olympiads.

Teachers' opinion about necessity of Math Olympiads

Although Math Olympiads in Latvia are very popular, achievements of students are quite low. To analyze this disparity, the Correspondence Mathematics School has organized a math teachers' survey about advanced math education system in Latvia.

In the survey took part 248 math teachers from all regions of Latvia, of various ages, both from rural schools and cities, distribution of respondents was similar to general population of math teachers in Latvia, therefore the data gathered from the survey could be applied to all teachers.

One of the questions in the questionnaire was "*Are math Olympiads necessary and why?*" All teachers answer with "*yes*" and main arguments mentioned in the answers were:

- the deepening of knowledge, the development of logical thinking, development of students personality, a wider vision of mathematics, motivation of students to learn more (18%);
- opportunity for students to compete in the higher level outside the schools, to meet like-minded, the competitive spirit can motivate to learn more (11%);
- it is an opportunity for gifted and talented students to show their abilities (as requirements of curriculum are reduced) (10%);
- to identify talented, gifted students and develop they further, to create new experts, scientists, clever people in Latvia (6%);
- to promote interest of students to mathematics, to draw attention of society to science (5%).

Besides these answers, the opinion that problems of competitions are very hard was quite frequently expressed. Some teachers (3%) had viewpoint that problems and tasks presented in math Olympiads create an aversion, disappointment and disgust to the mathematics for some good students.

Content of math Olympiads' problems

Form and organization of math Olympiads in Latvia stays unchanged from the beginnings. Olympiad's problem set consist of five problems (for each grade), the full solutions and explanations should be written. Time allowed for solving is 5 hours (300 minutes). Solution of each problem is marked by 0 – 10 points, maximum one student can get 50 points.

However, the content of the mathematical Olympiads problems has changed over time. At the beginnings there usually were school curriculum related problems, but maybe more technically complicated, mainly problems of continuous mathematics were represented. Along with the development of computer science problems of discrete nature and combinatorics became more popular. However in school curriculum main emphasis is still on the continuous mathematics (both algebra and geometry), therefore the gap between "school mathematics" and "Olympiad mathematics" arises. Also the role of proof in school tasks is decreasing during last decades while in Olympiads it is of the greatest importance. Standard school tasks usually are quite simple, done in one, two or three steps, usually on one topic, and of the lack of time very few number of more complicated problems are considered while in the Olympiads almost all problems are synthesis of various topics and branches of mathematics.

The main aim of the teachers' survey was to find out ways how to make the Olympiads' problem set more "friendly".to students. In the questionnaire was included question "*What are your suggestions for doing math Olympiad problems' set more accessible for students?*". Teachers' answers were (more than one answer could be marked):

- One problem from school curriculum (77%);
- One problem from previous year problem set with minor changes, maintaining the solution ideas (62%);
- One easy problem from some previously announced topic (35%);
- One problem from previous year problem set without changes (28%);
- One problem from last two years problem sets (18%).

Problem from previous year problem set (so called *repeated problem*) during last 20 years was used in problem sets of Open math Olympiad and regional round of state Olympiad. However there are different attitudes to *repeated* problem.

Those who advocate *repeated* problem refers to positive effect concerning the students' training process. As there is large amount of topics and methods could be appearing in Olympiad problems, students and teachers often get confused, while knowing there will be one problem (or similar) of five previous years' ones, they can start with studying them. Consequently, at least five problems and methods of solution will be studied.

In contrast, those who are against, indicate that *repeated* problem often shows memorizing capacity of a student not his/her solving skills. More about *repeated* problem is discussed in Kaibe&Rācene (2009).

Taking into account teachers' opinions, the jury of State Olympiad decided that starting with school year 2014/15 *repeated* problems will not be included into the Olympiad problems' set. It will be replaced by one problem from school curriculum and one problem from some previously announced topic. For year 2014/15 the announced topic is *Method of Invariants*. Announcement contains also short description of method and some examples of its usage in problem solving.

As results of abovementioned teachers' survey show, training process for mathematical Olympiads mainly is organized as previous years' Olympiads' problems' solving problems of previous years (78%), while only 11% respondents marked, that theory and examples on various topics are considered in extracurricular lessons. Therefore previously announced topic is of the greatest educational value – it motivates teachers' to organize lessons on specific topics as well it is opportunity for students to acquire more carefully at least one method/topic each school year.

METHOD OF INVARIANTS

General methods of reasoning

Prof. Agnis Andžāns, developer of mathematical Olympiads and advanced math education system in Latvia in his habilitated doctor's Thesis (Andžāns, 1995) gives description of elementary mathematics:

Elementary mathematics consists of: 1) the methods of reasoning recognized by a broad mathematical society as natural, not depending on any specific branch of mathematics and widely used in different parts of it, 2) the problems that can be solved by means of such methods.

Such general methods considered in (Andžāns, 1995) are:

- Mean value method and Pigeonhole principle,
- Method of invariants,
- Method of extremal element,
- Mathematical induction,
- Method of interpretation.

The studying of the abovementioned methods has also following positive general pedagogical effects:

- the demonstration of the unity of mathematics,
- aesthetically considerations,
- the possibility to use the underlying ideas of the methods outside mathematics,
- the broadening of the concept of proof. (Bonka & Andžāns, 2004).

Method of Invariants in Olympiads' problems

Method of Invariants was studied by Līga Ramāna, who defended her PhD thesis in 2004 (Ramāna, 2004). On the basis of her studies a teaching guide for school students and teachers was issued in Latvian, and in 2013 this brochure was translated in Lithuanian (Andžāns, Reihenova, Ramāna & Johannesson, 2013).

The method of invariants in Olympiad's problems is usually used in proofs of impossibility. The sense of method of Invariants is to find an invariant property of the given process and observe that final situation doesn't have this property.

The method of invariants is chosen as the *year's topic* 2014/15 because it is one of general method of reasoning and has also significant role in higher mathematics and calculus (e.g., invariance of form of differential, Euler's formula for planar graphs, etc.).

Further, problems on invariants will be considered, included into problems' sets of State Olympiad's regional round for high school students in year 2015.

9th grade. In each of the vertices of a regular octagon the numbers 7, 15, 3, 17, 1, 9, 5, 11 (in given order) are written. The following operations are allowed:

- choose some number and add both numbers written in the neighbour vertices;
- choose some number and subtract doubled number written in the opposite vertex, if the difference is positive.

Is it possible to get the number 2014 in some vertex by repeating allowed operations?

Answer: no; *invariant* – all numbers remains odd after each operation.

10th grade. Given the natural number 30, the following operations are allowed:

- add 6;
- divide by 4 if number is divisible by 4;
- arrange the digits of number in some order (avoid the first digit is 0).

Is it possible to get the number 2015 by repeating allowed operations?

Answer: no; *invariant* – all numbers are divisible by 3 after each operation.

11th grade. The numbers written in the cells as shown in the Figure 1 are given. It is allowed to choose any three cells that are arranged as shown in Figure 2 and add to these numbers +1 or -1. Is it possible, repeating allowed operations, to get in all cells the number 2015?

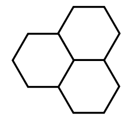

Figure 1. Figure 2.

Answer: no; *invariant* – sum of all numbers will be congruent 1 modulo 3 after each time operation repeated.

12th grade. The natural number 139 is given. The following operations are allowed:

- add to the number the sum of its digits;
- subtract from the number the sum of its digits.

Is it possible to get the number 193 repeating allowed operation?

Answer: no; *invariant* – the number is congruent to the sum of its digits modulo 9, therefore after at least one subtraction all numbers will be divisible by 9. The number 193 is not divisible by 9, therefore only first operation could be used. Considering the sequence of transformations, it easy to conclude impossibility.

First results of this Olympiad (~10% of all participants) shows that students are not yet well familiarized with the usage of the method of Invariant. These data are summarized in the table below.

Grade	Average result (max 10 points)	Correct solutions (9 – 10 points)
9th	5,4	39%
10th	2,8	10%
11th	2,4	10%
12th	4,7	22%

Table 1: Results in solving Invariants' problems in regional Olympiad 2015.

CONCLUSIONS

To achieve good results in math Olympiads it is necessary to work seriously before, acquainting various methods of solution and reasoning, as well as training in problem solving. To help students and teachers in that, jury of Latvian math Olympiads decided to announce one "*year's topic*". For this year announced topic is method of Invariants.

As this was the first year of the new regulations, the first results are not so well. It is because of not all students and teachers were well introduced to new rules and they didn't study proposed materials well. However, personal observation of the author are more optimistic – students who participated in lessons of math circles where this topic was considered, were able to solve these problems. Moreover some of them discerned other invariants, different from the ones identified within the official solution.

More careful analysis of the strengths and weaknesses of the "*year's topic*" idea could be done only after few years of its realization, but I believe, it will successful.

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LE-MATH: LEARNING MATHEMATICS THROUGH NEW COMMUNICATION FACTORS

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Abstract. *In this paper we introduce the guidelines for the two methods developed by the Le-MATH project, that is the MATHFactor and the MATHeatre methods. The guidelines are developed based on the collection and study of good practices in more than 10 European countries and they are available in 10 EU languages. Many pupils as well as parents consider mathematics to be a difficult and boring subject. Instead of studying mathematics (or other subjects) many students prefer to spend most of their time by watching TV, playing video games or on their mobile devices sending texts, pictures and videos. One way to bring pupils back to the "playing field of education" is to employ the use of similar tools - "weapons like the opponents", in order to communicate the learning of mathematics in a non-traditional way, like a game through theatre or competitions similar to the well-known X-Factor and other. The Le-Math project, funded by the European Commission and coordinated by the Cyprus Mathematical Academy with 12 participating partners, undertook the creation of these tools, in the period from November 2012 until its completion in October 2014.*

Key words: Mathematics, Learning, MATHFactor, MATHeatre, LE-Math, good-practices, guidelines

INTRODUCTION

Many students claim that mathematics is often too abstract and therefore difficult to understand. As a result, this project developed different and innovative approaches by inviting teachers and pupils together to apply new communication methods in the learning of mathematics, which could be fun and enjoyable at the same time. An approach, that brings new ideas in the context of "play and learn."

This European project developed a new methodology for the learning and teaching of mathematics to students aged between 9 and 18, which subsequently can be used in any school environment. It will also make learning more attractive and enjoyable for all students and it will strengthen their skills for creative thinking. These methods could be used in other subjects of the education curricula, as well as for other age groups.

The consortium comprises partners from universities, schools, mathematics associations, foundations, theatre schools, art schools and enterprises.

The project activities contribute to the Education and Training 2020 as it is enhancing creativity and innovation among youth. It also contributes to the benchmark for decreasing low-achievers in basic skills (mathematics and science) to 15%. It promotes the European Cooperation on schools in fundamental aptitudes, by supporting the key competence for mathematics.

GOOD PRACTICES IN THE EUROPEAN SPACE

In this work package we collected practices relevant to the subject and we developed them in an e-book. In this electronic manual one can find current or past activities. The final version of the manual can be found on the website of the project www.le-math.eu (click on the menu Guidebooks 2014-2015 and choose folder).

MATHEATRE METHOD

The Math Theatre follows the same rules of a normal theatrical play, but with the content of the play directly related to mathematics and with the actors being students between the age 9 and 18. It can have all the forms that characterize theatrical plays such as drama, comedy, musical etc. and the central plot can be based in any mathematics related subject from the school curriculum or from the history of mathematics. The difficulty of this activity lies in the fact that the dialogues of the actor-students must pass some mathematical knowledge to the audience. For supporting this part the project developed a Manual of Scripts for MATHeatre, so teachers and pupils can use in developing their own theatre play for communicating mathematical learning. The "MATHeatre Guidebook", is published on the project website, which contains the guidelines and the accompany tools. The electronic publication is presented in two different forms, one with the tools attached as links and one self-contained interactive book. A competition was launched through the project, for the writing of such plays and the submitted plays are published in the Manual of Scripts for MATHeatre. Furthermore, the project published theatre play dialogues in mathematics especially for the age group 9-12, called "Mathematical Stories for Theatre".

During the second year of the project, a European competition with international participation, titled MATHeatre EUROPE 2014 was launched. Schools, organizations or groups of students were eligible to participate, by applying the first draft of the guidelines published in September 2013 and preparing a play of a total duration of 5-12 minutes, with 2-10 participating actors. During the first phase of the competition (Sept. 2013-Feb. 2014), the participants had to upload their theatrical play on the Le-MATH platform. After the first evaluation process the best participants of two different age groups (9-13 and 14-18) were invited as finalists. The finals were held during the EUROMATH student conference on the 24-28 of April 2014 and the results are published on the project website.

The participants in the MATHeatre Competition and Evaluation were:

<u>MATHeatre EUROPE 2014</u>

AGE GROUP 9-13

PHASE I: Submissions: 14 teams

Number of Students: 80 students

Countries: Bulgaria, Czech Republic, France, Hungary, Italy, Romania

PHASE II Finalists: 6 teams

Number of Students: 37 students

Countries: Czech Republic, France, Italy, Romania

For the final competition in Cyprus, 5 teams with 31 students participated.

AGE GROUP 14-18

PHASE I: Submissions: 17 teams

Number of Students: 104 students

Countries: Bulgaria, Cyprus, France, Lebanon, Greece, Serbia, Slovenia

PHASE II Finalists: 8 teams

Number of Student: 52 students

Countries: Bulgaria, Cyprus, France, Lebanon, Slovenia

The evaluation criteria of the math theatre are published in two different forms; one for activities within the school environment and the other for open public competitions like MATHeatre Europe 2014. Evaluation criteria are flexible to be adapted to different education systems. The evaluation criteria for MATHeatre are shown in the next section below.

Assessment Criteria for MATHeatre

The criteria above can be used according to the education system of each country or school.

The assessment concerns:		Qualitative levels			
		Lower 5-6	**Interme diate 7-8**	**Higher 9-10**	**SCORE (Factor)**
1	**Mathematical content**				X
	Relevance of concept(s) discussed				
	Ability to make a mathematical theory comprehensible				
	Approach used to explain theoretical elements				
2	**Theatrical aspect**				Y
	Quality of expression: Delivery: speed of the speech (slow or fast) Volume: speech is loud enough to be understood. Articulation: clear pronunciation Vocabulary : richness of the vocabulary used				
	Space management and interaction				
	Respect of instructions : length: 5 minutes to prepare the stage, 5-12 minutes to play				
3	**Creativity of the staging**				Z
	Originality of the appearance and use of costumes				
	Use of the electronic back screen of the stage: originality of the projection on the screen and harmony with the play				
	Originality and appropriate use of sound effects and music, if needed				
	TOTAL X· Y· Z				

MATHFACTOR METHOD

The MATHFactor is an individual activity of communication related to mathematics, in the sense that a student will have to prepare and explain within a short time of 3 minutes, mathematical concepts, theorems, applications, or aspects of the history of mathematics etc., in a simplified manner so they can be understood by non-experts or students of same age. During the presentation the use of interactive projection tools and the blackboard is not permitted, but the student may use small visual objects that can be carried using one hand.

A good presentation will be evaluated based on the high articulation of the participant and his/her ability to impart knowledge to the audience, the presentation of mathematical concepts, for its content, its innovative approach in presentation and the talent exhibited to the viewer.

The whole approach it is based on the well-known TV game X-Factor, but it is centered on mathematics instead of singing. This method could be used as an educational activity within the classroom and/or in open public competitions.

The evaluation criteria for MATHeatre are shown in the next section below.

The assessment concerns:		Qualitative levels			
		Lower 5-6 points	Intermediate 7-8 points	Higher 9-10 points	SCORE (Factor)
1	**CONTENT**				X
	The degree to which the student demonstrates understanding of mathematical concepts and relationships between these	Displays basic knowledge	Displays good knowledge	Displays excellent knowledge	
	The quality of the student's analysis, conclusions and reflections, as well as other forms of mathematical reasoning	Uses some substantiated reasoning to make the mathematics understanding easy	Uses acceptable mathematical reasoning to make mathematics understanding quite easy	Uses excellent mathematical reasoning to make mathematics understanding very easy and almost obvious	
2	**CLARITY**				Y
	The quality of the communication. How well the student uses mathematical expressions (language and representation)	Expresses him-/herself simply, but understandably, using a mathematical language and approach suitable for the topic and non-expert audience	Expresses him-/herself clearly using a mathematical language and approach suitable for the topic and non-expert audience	Expresses him-/herself very clearly and confidently using a mathematical language suitable for the topic and non-expert audience	
3	**CHARISMA**				Z
		Displays some adaptation to the audience, e.g. by looking up, speaking clearly and/or showing commitment.	Displays relatively good adaptation to the audience by looking up, speaking clearly and presenting facts in an interesting or engaging way. Presentation and body language that causes impression the audience.	Displays good adaptation to the audience by looking up, speaking clearly and presenting facts in an interesting and engaging way. Presentation and body language that causes impression and excitement to the audience	
				TOTAL X· Y· Z	

EXPERIMENTATION AND EVALUATION

The experimentation and evaluation process took place in different phases and levels.

MATHeatre EUROPE 2014

MATHFactor EUROPE 2014

The whole effort was based on an international level competition. Participants were divided into two different age groups (9-13 and 14-18), in order to better serve the overall aim of the project and in order to give the necessary incentives and spark the interest of both students and teachers. The first phase of the competition opened on September 2013 and closed on February 7, 2014, with the participation possible via online submissions. After evaluating of the online first phase the finalists were invited to participate in the live international finals which took place during EUROMATH 2014.

During this process, the involvement and the activities of the students were evaluated, as well as the role and the impressions of the teachers that supported the effort. Additionally, their comments and remarks regarding the first draft of the guidelines were taken into account. The international finals were also assessed and the results are used for improving the procedures in 2015 as well as supporting sustainability. Additionally, the results were used for improving the guidebooks for the MATHFactor and the MATHTeatre methods as well as the transparency of the procedures.

<u>MATHFactor EUROPE 2014</u>
AGE GROUP 9-13
PHASE I Submissions: 4 students
Countries: Cyprus, Czech Republic
PHASE II Finalists: 4 students
Countries: Cyprus, Czech Republic
All students were invited to participate in the final competition in Cyprus, as this number was low.
AGE GROUP 14-18
PHASE I Submissions: 24 students
Countries: Bulgaria, Croatia, Cyprus, Greece, Hungary, Iran, Romania, Serbia, Slovenia, Spain, Sweden
PHASE II Finalists: 7 students
Countries: Bulgaria, Croatia, Cyprus, Greece, Serbia, Slovenia
All students participated in the final competition in Cyprus on 26 April 2014 at 17.00 at Hilton Cyprus.
The evaluation report is published on the project's website in the listing of outcomes (http://www.le-math.eu/assets/files/Work%20Package%204_REPORT31%2010%202014.pdf).

An important part of the project's sustainability is also the creation of a five-day training programme for teachers, which is offered as a training course open for participation through funding provided by the ERASMUS+ KA2 programme managed by the ERASMUS+ National Agencies of the programme countries. The course outline and learning outcomes can be read in (http://www.le-math.eu/index.php?id=528). The first session is organized on 25-31 March in Athens, in parallel to EUROMATH 2015 conference.

LEARNING AND EXPLOITATION

The Evaluation Report presented on http://www.le-math.eu/assets/files/Work% 20 Package%204 REPORT31%2010%202014.pdf, provides evidence and analysis of the positive impact that the method had on students attitudes and improved performance. From September 2014 the project Le-MATH published the final version of the Guidebooks 2014-2015 and the competitions MATHeatre Europe and MATHFactor Europe 2015 inviting teachers from all over Europe and beyond to apply the methods and use the competitions as incentives to attract the interest of their students. Letters to all Ministers of Education in European countries and beyond are sent inviting the Ministries to support local regional or national competitions using the fact that winners of such competitions earn a place to the finalists of the final international competitions. Those interested to apply directly are invited to participate in the two Phases procedure through the Le-MATH platform. Phase I is the online submission of their video and if approved to be invited to the finals. The international finals is held during the EUROMATH 2015 student conference on the 25-30 March 2015 (www.euromath.org).

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* Coordinating Organization of the project is the Cyprus Mathematical Society (CY-Gr. Makrides, A. Philippou, C. Papayiannis) along with 12 partners from Cyprus, Greece, Bulgaria, Romania, Austria, Sweden, France, Spain, Czech Republic, Belgium and Hungary. The cooperating bodies are **Thales Foundation of Cyprus** (CY-A. Skotinos, P. kenderov, E. Christou, L. Zeniou-Papa, C. Christou), **Charles University in Prague-Faculty of Education** (CZ-J. Novotna, A. Jancarik, K. Jancarikova, J. Machalikova), **Loidl-Art** (AT-H. Loidl), **VUZF University** (BG-S. Grozdev), "**CALISTRAT HOGAS" National College Piatra-Neamt** (RO-N. Circu, L-M Filimon), **Lyckeskolan** (SE-M. Manfjard Lydell), **LEOLAB** (ES-M. Munoz, B. Dieste), **Junior Mathematical Society Miskolc** (HU-P. Kortesi), **European Office of Cypru**s (BE-CY-R. Strevinioti, D. Tsikoudi, C. Katsalis), **Collège Saint Charles** (FR- K. Treguer, E. Gueguen, **E. Darees**), **National Technical University of Athens, Institute of Communication and Computer Systems** (GR- K. Karpouzis, A. Christodoulou), **Com2go Ltd** (CY-G. Economides, N. Nirou, V. Cheminkov).

ORAL PRESENTATIONS 1.3.

Problem solving and problem posing situations

Chair of the session: **Ewa Swoboda**

PROBLEM SOLVING AND CHOICE-BASED PEDAGOGIES

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Abstract. *An important premise of creative thinking is that a problem solver is in position to choose. The literature on open-ended problem situations focuses on the mechanisms underlying learner choices of problem-solving moves. This article argues that there is room for empowering students to make choices of additional types: choosing an extent of collaboration, choosing a mode of interaction, and choosing an agent to learn from while solving a challenging problem. Theoretical argument is supported by an empirical example drawn from an on-going study on the interaction between cognition and affect in high-school students' long-term geometry problem solving.*

Key words: choice-based pedagogies, collaborative problem solving, online discussion forums, shifts of attention, Olympiad geometry

INTRODUCTION

Since the time of *Science and Method* by Poincaré (1908/1948), it is broadly accepted that choice is an important constituent of mathematical creativity (e.g., Sriraman, 2004)[1]. A corollary of this idea is that pedagogies supporting mathematical creativity must include for individuals opportunities to choose. Engaging students in challenging open-ended problem situations – I use this term in a broad meaning assigned to it by Cifarelli and Cai (2005)[2] – is probably the most widespread type of pedagogy aimed at empowering students to make choices while doing mathematics. Research on open-ended problem situations informs us about some of the mechanisms underlying learners' solution moves or, more generally, about their choices related to the use of mathematical knowledge and strategies (e.g., Mamona-Downs & Downs, 2005). Choices of additional types are rarely left to the students. In particular, the choice of a problem to be solved or the choice of an instructional setting, in which problem solving takes place, are, as a rule, made by a teacher. In particular, students are normally expected to conform to the teacher's decisions about whether to approach a problem individually or with the classmates and how much time to devote to the problem. Given the complexity involved in making such decisions for a teacher (e.g., Schoenfeld, 2010), it is reasonable to suggest that any teacher choice is likely to be not optimal for at least some of the students.

There is an alternative, however. Its roots can be traced to seminal work of Dewey (1938/1963), who substantiated the idea that students must be involved in choosing what they learn and how. Nowadays, creativity-supporting pedagogies that provide students with opportunities for self-directed learning, including the opportunities to choose suitable for them modes of learning, exist. Some of them are called *Choice-Based Pedagogies*[3] (hereafter, CBPs). CBPs are flourishing in business schools and in art education (e.g., CBAE, 2008,

[1] In particular, such indicators of creativity as flexibility and fluency are functions of choices available to a problem solver (e.g. Koichu & Orey, 2010).

[2] Cifarelli and Cai (2005) define open-ended problem situations as situations, in which "some aspect of the task is unspecified and requires that the solver re-formulate the problem statement in order to develop solution activity." (p. 302). Note that problem solving, problem posing, various exploration activities, model-eliciting activities are all within the range of this definition.

[3] Note the difference between Choice-Based Pedagogy and Pedagogy of Choice notions. The latter notion is usually used as a name of pedagogies, in which *teachers'* choices are considered (Cummins, 2009).

Douglas & Jaquith, 2009), but are still not common in mathematics education involving open-ended problem solving.

This theoretical article aims at substantiating the following position: CBPs are feasible and can be useful for creativity development in the context of open-ended mathematical problem situations. Specifically, I stop on the following choices: a student's choice of an extent of collaboration, a mode of interaction, and an agent to learn from when solving a challenging mathematical problem. The next section outlines an emergent framework of mathematical problem solving, on which the development of CBPs in mathematics education can rely. It is followed by the section about characteristics of CBPs in mathematical open-ended problem situations. Then an example is presented. It is drawn from an on-going study on the interaction between cognition and affect in students' long-term geometry problem solving supported by online discussion forums.

CHOICE IN THE CONTEXT OF AN EMERGENT FRAMEWORK OF PROBLEM SOLVING

Poincaré (1908/1948) describes a creative problem-solving process as a multi-stage pathway of conscious and unconscious steps towards the minimalistic choice of a "proper" combination of ideas out of a huge number of possible combinations. This metaphoric description of creativity is still appealing though, as Sriraman (2004) noted, it seems to overlook the issue of novelty and the role of social interactions. In search for a less metaphoric and more operational (for analyzing empirical data on problem-solving processes) theory that would be consonant with the idea of choice, I arrived at Mason's theory of shifts of attention. This theory had initially been formulated as a conceptual tool to dismantle constructing abstractions (Mason, 1989) and then extended to the phenomena of mathematical thinking and learning (Mason, 2008, 2010). Palatnik and Koichu (2014) adopted this theory as a tool for analyzing insight problem solving. Recently, it has been utilized as a foundation of a confluence framework of problem solving in different educational contexts (Koichu, 2015)[4].

In terms of the framework, the process of inventing a solution idea is considered as a pathway of a solver's shifts of attention, in which objects embedded in the problem formulation and problem situation image are attended to and mentally manipulated by applying available schemata. The process at large is goal-directed, but particular shifts can be sporadic. (Note a connection to Poincaré's conscious and unconscious "choices"). A pathway of the shifts of attention depends on various factors, including: the solver's traits, his or her mathematical, cognitive and affective resources (cf. Schoenfeld, 1985) and the context in which problem solving takes place. The framework embraces three sources from which one's pathway of the shifts of attention can stem while solving a particular problem: (i) individual effort and resources, (ii) interaction with peer solvers who do not know the solution and struggle in their own ways with the problem or attempt to solve it together, and (iii) interaction with a source of knowledge about the solution, such as a teacher or a classmate who has already found the solution but is not yet disclosing it. These sources can be employed in separation or complement each other in one's problem solving.

Each source has its potential benefits and limitations from an individual problem solver's perspective. Individual resources can suffice or not for solving a problem. In the latter case

[4] The next paragraph is an abridged version of the description that appears in Koichu (2015).

students, including the gifted, prefer to interact, but not all the time (Diezmann & Watters, 2001). Interactions with peers can help a solver to focus his or her attention on a "proper" object or combination, but can also be overwhelming or misleading. Indeed, when nobody in a group knows how to solve the problem, the other members' inputs of potential value are frequently undistinguishable for the solver from the inputs of no value. Consequently, it can become too effortful for the solver to follow and evaluate the inputs of the others, and he or she can prefer working alone on with a source of knowledge about the solution.

The latter case can be seen in a widespread situation when a teacher, who knows the solution, "orchestrates" the discussion by responding more positively to some of the students' ideas than to the others. In this way, the problem is usually solved before the bell rings, which is important in a regular classroom setting. This situation, however, bears for a solver a danger of being deprived from inventing the solution or being misled by a deceptive feeling "we solved the problem with the teacher, so next time I will be able to do so alone." The feeling is deceptive because when a source of knowledge about the solution is present but does not tell the solution, the solver can attempt extracting the solution from the source (e.g., Koichu & Harel, 2007) instead of persisting in problem-solving search.

The point is that when a problem-solving process in an educational context is considered at the level of students' shifts of attention, it becomes visible how overwhelming the diversity of individual pathways and possible scenarios of interactions is. In turn, it becomes clear how immensely difficult it is for the teacher to choose for his or her students types of interactions that would suit their individual problem-solving needs and traits. The difficulty shows even when a skillful teacher manages to combine individual work, work in small groups and a whole-class discussion in a mathematics lesson.

CHOICE-BASED PEDAGOGIES SUPPORTING OPEN-ENDED PROBLEM SITUATIONS

As mentioned, the very nature of open-ended problem situations presumes that students are required to specify some aspects of given tasks and make solution choices. Proponents of CBPs in the field of art education go beyond this. Their central premise is: "The student is an artist…In an authentic choice-based environment, students have control over subject matter, materials, and approach" (CBAE, 2008, p. 6). A classroom functions as a studio with different activity centers working in parallel, and students make "real choices" about in which activity to take part and how (Douglas & Jaquith, 2009).

Transposition of this approach into the context of mathematical problem solving might be as follows. The student is a problem-solver, who is in position to choose who, when and how to interact with when solving the problem[5]. Problem solving occurs in a studio-like learning environment, in which different activities and discourses take place, sometimes in parallel. In particular, each student at different times chooses the most appropriate to him or her:

- extent of collaboration, from being actively involved in exploratory discourse with peers of his or her choice to being an independent problem solver;
- modes of interactions, that is, whether to talk, listen or be temporary disengaged from the collective discourse, as well as whether to be a proposer of a problem-solving idea, a responder to ideas by the others or a silent observer;

[5] Sometimes a student can be involved also in choosing a problem to solve. The potential of this possibility is considered, for instance, in Ph.D. study by Palatnik (in progress).

- agents to learn from, that is, a student can decide whose and which ideas are worthwhile his or her attention, if at all.

It is not easy to imagine how to practically realize the outlined choice-based environment in a regular time-constraint mathematics classroom setting; an image of chaos comes to the mind's eyes. As mentioned, CBPs can provide an alternative.

EXAMPLE OF A CBP: PROBLEM SOLVING IN AN ONLINE DISCUSSION FORUM

A 10th grade of 16 regular (that is, not identified as gifted) students and their teacher took part in an experiment, in which challenging Olympiad-style geometry problems were given for solution during 5-7 days each, in an environment combining classroom work and work from home. The work from home was supported by an online discussion forum at Google+. Realization that this learning environment is a CBP came to me during the analysis of the data, which were collected from the following sources: content of the forums, the teacher diary, interviews with selected students, and reflective questionnaires that have been filled in by all the students after working on each problem.

The students indicated in the reflective questionnaires that they had worked collaboratively for about 40% of time that had been devoted to the problems. (On average, the students worked on each problem for overall of 3 hours distributed during 1-3 days). As a rule, the students chose to collaboratively work in the forum when they were stuck and sought for new ideas or for the feedback on their incomplete ideas.

The exposition below focuses on one student, Marsha[6], who solved one of the problems in a particularly original way. It is of note that, according to the teacher, Marsha was not an active student in a classroom or a successful student in terms of mathematics tests. The problem (Sharygin & Gordin, 2001, No. 3463, see a drawing in Figure 1a), was as follows:

Two extrinsic circles are given. From the center of each circle two tangent segments to another circle are constructed. Prove that the obtained chords (GH and EF – see the drawing) are equal.

The challenge was that the problem could hardly be solved by including the chords in some pair of congruent triangles or in a parallelogram. The intended solution was based on consideration of two pairs of similar triangles, $\Delta MEP \sim \Delta\, MCN$ and $\Delta NGQ \sim \Delta\, NAM$ (see Figure 1b). $EP = \frac{r \cdot R}{MN}$ from the first similarity, and $GQ = \frac{r \cdot R}{MN}$ from the second similarity, which concludes the proof. Two solutions based on this idea were posted by two students by the end of the forum, two days after Marsha's solution.

The key idea of Marsha's solution was that $KL \parallel EF \parallel GH$, and this was particularly difficult for her to prove. Her proof of this fact consisted of 24 "claim - justification" rows. She then used this fact in order to prove similarity of two pairs of triangles, $\Delta KNL \sim \Delta\, GNH$ and $\Delta MKL \sim \Delta\, MEF$. She concluded the proof by consideration of proportions stemming from these similarities, in conjunction to a proportion stemming from a "bridging" pair of similar triangles, $\Delta MAK \sim \Delta\, NCK$. The actual problem-solving process was not straightforward at all. (I plan to present it in some detail at the conference.)

[6] All the names are pseudonyms.

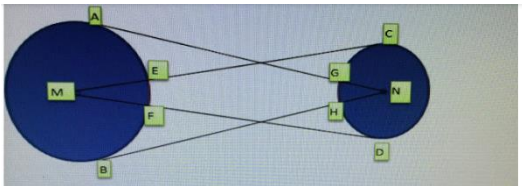

Figure 1a: The given drawing

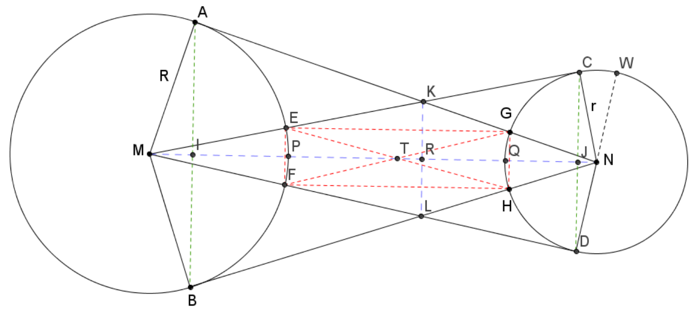

Figure 1b: Summary of auxiliary constructions

Figure 1: Drawings for the two-circle problem (Sharygin & Gordin, 2001)

In brief, at the beginning Marsha focused her attention on how to use MN and the radiuses of the circles. She worked on this idea for some time with one of the classmates, Sarah. The data show when Marsha and Sarah's pathways of the shifts of attention departed and why. Marsha then attempted to prove that $EGHF$ was a rectangle. This attempt for some time was in line with the reasoning of another participant, Mary. After exchange of the ideas and a period of being silent at the forum, Marsha announced that she completely changed the direction and that she is "several moments from the proof". However, her initial explanation of the proof at the forum had logical flaws. Marsha succeeded to produce a mathematically valid proof after polishing her reasoning in reply to clarification questions by two other participants of the forum.

Marsha was very active at the forum: 62 out of 192 verbal entries to the forum belong to her. However, Marsha indicated in her questionnaire that she worked alone for about three out of five hours that she devoted to solving the problem. It was also observed that Marsha had been highly selective when choosing with whom and how to communicate.

The point is that the described learning environment, in which Marsha worked creatively and succeeded, was indeed a choice-based environment for her: she was in position to choose (and change when needed) an extent of collaboration, a mode of interaction, and an agent to learn from. *Q.E.D.*

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PROBLEM POSING COGNITIVE STYLE - CAN IT BE USED TO ASSESS MATHEMATICAL CREATIVITY?

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Abstract. *We asked students-prospective mathematics teachers in their last year of a master program to pose problems starting from a context rich in geometrical properties. We identified cognitive styles in geometry problem posing by analyzing the products (i.e. the posed problems) using three criteria of analysis: the validity of the students' posed problems (in terms of coherence and mathematical consistency); and two criteria that detect personal manners in addressing the task: metric geometry vs. "qualitative" geometry approach, and the degree of dependence of the posed problems. Both qualitative versus metric and dependent versus independent criteria are bipolar, which is relevant for discussing cognitive styles. By questioning which of the identified cognitive styles shows a creative behavior, we found that the development of independent and qualitative geometry problems is a better candidate for detecting cognitive variety and changes in cognitive frame, therefore, for cognitive flexibility.*

Key words: problem posing, creativity, cognitive flexibility, cognitive style.

INTRODUCTION

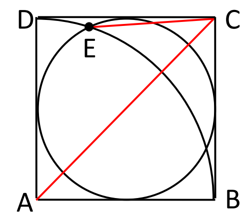
The development of mathematical creativity is considered both as a means and a goal in education. In research, efforts were often oriented towards defining it and proposing frameworks for its assessment, especially through problem posing and solving activities. Some researchers (e.g. Jay and Perkins, 1997; Silver, 1994) claim that problem posing may stimulate creativity, possibly even more than problem solving. There is no consensus, concerning the definition of mathematical creativity and its framework of study. For problem posing context, Kontorovich and Koichu suggested a framework based on four "facets": resources, heuristics, aptness, and social context in which problem posing occurs (Kontorovich & Koichu, 2009). A refinement of this framework integrates task organization, knowledge base, problem posing heuristics and schemes, group dynamics and interactions, and individual considerations of aptness as parameters in analyzing creativity in problem posing situation (Kontorovich, Koichu, Leikin & Berman, 2012).

Rather than looking for a summative indicator, in this article we propose the construct of Geometry-problem-posing cognitive style as an expression of both personal traits and creativity. Specifically, we were interested to answer the following questions: What kind of tool could provide information about mathematical creativity of university students in problem-posing contexts? How could we assess creativity in this case? We consider the framework that relies on the construct of cognitive flexibility, characterized by cognitive novelty, cognitive variety, and changes in cognitive framing (Pelczer, Singer & Voica, 2013; Voica & Singer, 2011, 2013) offering a good context to operationalize the study of mathematical creativity in a PP context, through what was called problem posing cognitive style. In general, the cognitive style is seen as an individual approach to organizing and representing information (Riding & Al-Sanabani 1998). Crespo (2003) uses the term "problem posing style" in relation to proposers' views and beliefs on mathematics problems.

METHODOLOGY

Context and sample

During a course in Math Education, we asked students-prospective mathematics teachers in their last year of a master program to pose problems starting from a context rich in geometrical properties.



We presented students with the next figure and its descriptive construction:

The figure contains: the square ABCD, the circle inscribed in this square, and the circular arch of center A and radius AB.

To make students familiar with the given figure, but also to show that this figure offers a rich geometrical context, the teacher indicated an example of a problem emerging from it:

Taking into account the above hypotheses and notations, prove that $2 \cdot EC = AC$.

We asked students to pose as many as possible problems related to the above figure, within the next three-week time. The students got the assignment to write the posed problems in the order in which they emerged from their minds, if possible, and to add a proof, or at least an indication of solving, for each posed problem. In all, six female students (out of the eighteen students enrolled in this course) responded to the task. These students represent our sample.

Criteria for data analysis

To identify styles in geometry problem posing, we analyzed the products (i.e. the posed problems) using three criteria of analysis, targeting respectively: the validity of the obtained product; types of operating in synthetic geometry; and the ”affiliation” relationship among the generated problems. Regarding validity, the students’ posed problems were analyzed in terms of their *coherence* and *mathematical consistency.* For this criterion, we have adapted the definitions used by Singer and Voica (2013) to the situation described in this paper. We further detail the other two criteria. Given that the starting problem provided a context of synthetic geometry, students have built their derived problems by choosing either a metric perspective or that of "qualitative" geometry based on congruence. A problem has been classified as being *metric* if its statement required finding sizes (for example, the area of a triangle) and the solving went through utilizing specific computing formula. The problems based on geometrical reasoning by using congruence without appealing to metric calculations were classified as *qualitative*. The third criterion was the degree of dependence of the posed problems. When a student managed to use a consistent part of a problem (in terms of data and results) in posing and solving another one, we considered that the second problem is *dependent* on the previous one. The identification of the dependence relationship was facilitated by the students’ display of their posed problems – in the temporal order in which they have been produced. A problem was considered *independent* when it has not been shown dependent from the previously posed problems, according to the ordered list provided by the student.

Both *qualitative versus metric* and *dependent versus independent* criteria are bipolar, and we represented students’ strategies to pose problems using a procedure similar to that of Gregorc for the cognitive style delineator (e.g. Gregorc, 1982).

RESULTS

Six students (*A., L., D., O., S.,* and *I.*) responded to the task, generating a total of 148 problems. Each of these students proposed at least 15 problems, and there is a student (*A.*) who generated 50 problems. We present three of the posed problems in Table 1, to get an image on the nature of students' proposals.

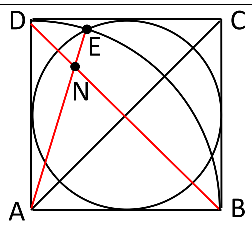
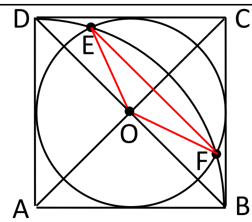
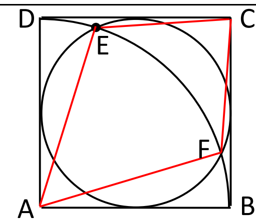
Problem 1 (posed by D.)	Problem 2 (posed by L.)	Problem 3 (posed by O.)
		
AN = 4/5 AB	*Prove that the area of the triangle EOF is an irrational number.*	*F is the symmetric of E with respect to AC. Show that AECF is not a trapezoid.*

Table 1. Three examples excerpted from the students' posed problems

We organize the results based on the three criteria presented above. In total, of the 148 generated problems, 130 (i.e. 88%) were coherent; in the rest of 12%, the new introduced elements were only denoted on the figure, but the text of the posed problem did not mention the introduction of these elements (such as, for example, in Problem 1 from above). In these problems, it was not sufficiently clear what the hypothesis is and what the consequences are; therefore, we classified these problems as being not coherent.

13 of the 148 posed problems were identified as being inconsistent, a large majority –91%, being consistent. As an example of inconsistent problem, we show problem 2 of Table 1. The accuracy of this problem depends on the chosen measurement unit (by which one can express the sides of the square ABCD), in other words, the mathematical model associated with this problem is inconsistent.

The 148 students' posed problems have a balanced distribution as referring to the *qualitative versus metric* criterion. The balance is no more preserved in terms of the degree of *independence*: 52 problems have been classified as independent, the remaining 96 being dependent. A graph representation of the problems distribution according to the three criteria is shown in Figure 1.

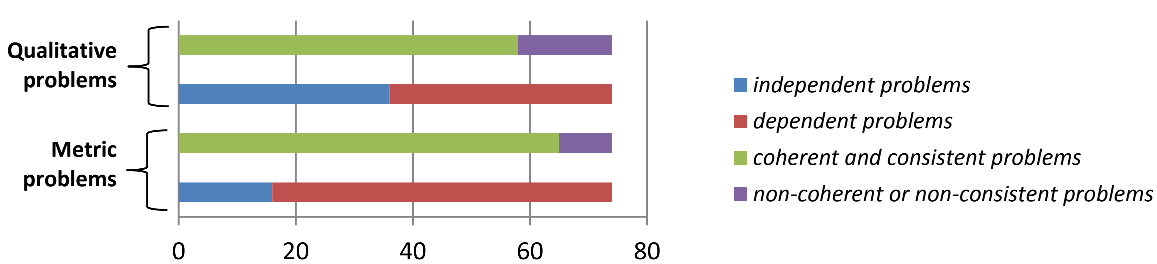


Figure 1. The posed-problem distribution according to three criteria

DISCUSSION

The diagram in Figure 1 indicates the situation of the 148 problems posed by the 6 students in relation to the three considered criteria.

The coherence-consistency criterion relates to the validity of the final product. Getting valid problems is a target pursued by all students in achieving their task. Indeed, most posed problems were coherent and consistent, and we did not record significant differences between students, based on this criterion.

We moved to an individualized (per student) analysis referred to the other two criteria. Given the bipolar nature of those criteria, characterized by the parameters: metric versus qualitative, and independent versus dependent, we graphed the distribution of each student's posed problems through a diagram represented in a Cartesian coordinate system. In this representation, the numbers on each axis, which indicates the percentage of problems that satisfy the respective parameter, determines a quadrilateral. The diagrams corresponding to three of the participants are shown in Figure 2. We note that the obtained quadrilaterals have different shapes and positions in relation to the axes, which shows significant differences between participants. The question we ask now is what significance these differences might have in terms of mathematical creativity.

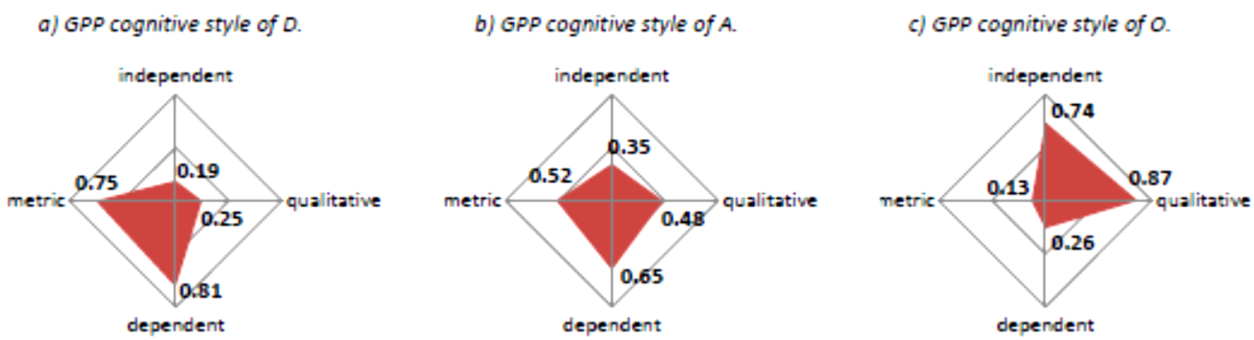


Figure 2. Diagrams representing the geometry-cognitive styles of three participants

The different shapes of the diagrams of Figure 2 shows that the two criteria detects personal manners in addressing the task, not imposed by the constraints of correctly solving the task. In 4 of the 6 analyzed cases, there is a relatively central position of the quadrilateral in the diagram, as in Figure 2b, indicating a relatively balanced distribution of values for the two criteria. The eccentric positioning of the quadrilaterals in Figures *2a* and *2c* show strong preferences in approaching geometry problem posing tasks: while *D.* prefers metric problems that "flow" one into each other, *O.* usually generates qualitative and independent problems. The students' preferences in PP, evidenced by these charts, may be put in relation to a particular cognitive style in problem posing; we will use the term *geometry-problem-posing cognitive style* (GPPCS) for describing individual approaches in this specific context. Therefore, the flowcharts of Figure 2 highlight the existence of different *GPPCSs* of students in our sample.

We further question which of these cognitive styles show a creative behavior. We analyse mathematical creativity based on a cognitive-flexibility framework, highlighting students' behaviors on three components: cognitive novelty, cognitive variety, and changes in cognitive framing (Singer & Voica, 2013; Voica & Singer, 2013).

The frequent use of dependent problems indicates the stability of the mental configuration induced by the initial geometric context, showing the mastering of the cognitive frame and fluidity inside the frame. The independence relationship invites the proposer to generate a greater number of problems. Thus, student *A*. proposed 50 problems, 32 of them being classified as dependent, and student *D.* generated 33 problems, most of these problems (27 from 33) being dependent. *A.* and *D.* seem to have a "cascade" approach: most of their problems are related, and many are reformulations of previous ones.

We further analyse the case of *O*; most of *O.*'s problems are independent. It seems that the development of independent problems is a better candidate for detecting changes in cognitive frame. Looking at her proposals, we can see that *O.* proposed 15 problems using different strategies: she "catches" an idea, formulates a problem, and then she looks for other properties of the given geometric configuration, extending the frame. Therefore, she shows capacity of making changes in her cognitive frame of the given problem context.

For cognitive variety, the indicator is the number of diferent posed problems. *A.* posed 50 problems, among which aprox. 65% are dependent problems, and *D.*, 33, among which 81% are dependent. Many of the problems proposed by *D*. (25 out of 33) and by *A*. (26 out of 50) are of metric nature. In general, measurement/measure is a productive procedural framework, thus explaining the big number of generated problems. In contrast, the independence and qualitative dimensions of the posed problems of *O.* indicates that the proposer has managed to generate *different* new problems, while the problems generated by *D.* and by *A.* were, in many cases, equivalent reformulations of the source problems.

What about cognitive novelty? Previous research (e.g. Voica & Singer, 2013) shows that, in the ages of 10-16 years, mathematical creativity manifests through incremental changes of a single parameter in problem-modification situations. The conclusion of this study was that in PP situations, cognitive novelty occurs only to a minimal level at age 10-16 years. In this paper, we found similar behavior, although the age and level of training are different in this new sample (23 to 24 years, students in a graduate program): relatively many of the posed problems contain obvious questions (immediate reformulations of previous problems or of well known results). Therefore, from this point of view, there is not a significant difference between the samples. We found that, although they are master students in their last year of graduate studies, they behave as novices in the problem posing context (for example, they ignore the symmetry of the initial configuration – which may generate thinking out-of-the-box; in general, they fail to have a global vision of the initial configuration). It seems that this limit of the student's expertise in our sample prevents them to be more creative: they do not have mechanisms of organizing information that allow them to generate "spectacular" problems far from the starting example.

CONCLUSIONS

In this paper we investigated the problems posed in a given geometric context by a group of students-prospective mathematics teachers enrolled in a master program. We identified different types of cognitive styles, which we sought to illustrate above. Thus, we found students whose problem posing cognitive style has a dominant focus on metric and dependent dimensions, while others' problem posing cognitive style has a dominant focus on independent and qualitative dimensions.

By questioning which of the identified cognitive styles shows a creative behavior, we found that the development of independent and qualitative geometry problems is a better candidate for detecting cognitive variety and changes in cognitive frame, therefore, for cognitive flexibility.

The above observations led us to conclusions that are directly correlated to the graphical description we provided for representing the GPPCSs: the positioning of the quadrilateral in the first quadrant (qualitative and independent) may indicate a higher level of creative capacity. In other words, the shape of the diagram can be a good predictor of students' mathematical creativity in geometry problem posing situations.

Some limitations of our study should be taken into account. On the one hand, we cannot draw firmer conclusions due to the small number of participants. On the other hand, the degree of creativity in PP may depend on the expertise level of participants. We intend to apply similar tasks with a larger sample. Until then, we can conclude that at least students' GPP Cognitive Style charts reveal different ways of students' knowledge organization and emergence: further research will show whether this is sustained by converging evidence.

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CREATIVITY AND BISOCIATION

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Abstract. *Creativity research has its roots in reflections upon eminent mathematics creating original and useful mathematical products. Its extension into the mathematics classroom leaves open many definitions and understandings of its value and role in education of gifted students. The extension of creativity to ordinary students who do not see themselves as gifted or the democratization of creativity is explored through the lens of Koestler's theory of bisociation and the relationship between bisociation and reflective abstraction the mechanism of learning proposed by Piaget is put forth.*

Key words: bisociation, reflective abstraction, incubation, illumination

HISTORICAL DEVELOPMENT OF CREATIVE RESEARCH

The historical development of creativity is typically traced to the original and creative work of eminent mathematicians. Inherent in this view is measurability through the end result or the product definition (Liljedahl,2009, 2013) in which the community of mathematicians is to verify the truth of as well as to judge the usefulness and originality of the idea presented.

The extension of creativity to the mathematics classroom is often traced to the work of Jaques Hadamard who synthesized the stages the Gestalt psychology Wallas to explain the setting in which the "Aha" moment occurs through a process-stage that begins with incubation in which the solver tries diligently but unsuccessfully to obtain the goal to the instantaneous insight of the illumination stage. Hadamard it would appear believed that the distinction between the creative genius and a creative individual learning mathematics that is known to the community but not him or herself is only a matter of degree. Yet research on incubation and the transition to illumination is scarce in mathematics education and the mechanisms poorly understood (Jutter & Sriraman, 2011).

Perhaps one reason is the view that concepts of incubation and illumination are considered, "…arachaic Gestalt constructs…" (Juter & Sriramen, 2011), which as Piaget (1989) and Koestler (1964) would note has fallen from fashion due in part to its overly strict focus on not only an instantaneous but also complete insight into problems and structures that in cognitive development takes places in repeated flashes of insight or eureka moments that build on one another and occur as a result of continued effort by the individual learner.

Another related reason is that the affective component of creativity is too difficult a concept for educational researchers and psychologists to measure objectively, indeed the intuitive nature of the creative experience is often tied to a self-transcendent force, which inspires a spontaneous "Aha" or Eureka moments simultaneously with a complete solution to the problem.

Transition from genius to gifted students

The transition from the so-called genius understanding of creativity, to a more inclusive definition and understanding that allows for research into creativity in the mathematics classroom especially in regards to research in problem solving (Silver, 1997) that has spurred a significant amount of research and a multitude of definitions (Mann, 2005).

One might also point out that definitions and understandings used to measure creativity with gifted students within the mathematical classroom can be viewed as being developed to distinguish between gifted and non-gifted or ordinary students.

Leikin (2009b) notes two formulations of creativity in mathematics educational research have been used to assess an individual's propensity for creativity in the mathematics classroom or student giftedness. The first is the ability for convergent and divergent thinking due to J. Guilford. "Convergent thinking involves aiming for a single correct solution to a problem, whereas divergent thinking involves the creative generation of multiple answers to a problem or phenomena, and is described more frequently as flexible thinking." Her review also notes the definition suggested E. Torrance i.e. the capacity of an individual for flexibility, fluency, novelty and elaboration. "Fluency refers to the continuity of ideas, flow of associations, and use of basic and universal knowledge. Flexibility is associated with changing ideas, approaching a problem in various ways, and producing a variety of solutions. Novelty is characterized by a unique way of thinking and unique products of a mental or artistic activity. Elaboration refers to the ability to describe, illuminate, and generalize ideas." (p.129).

Transition from gifted to ordinary students

Yet these characteristics such as fluency with mathematical content, the ability to view and understand various different approaches to a problem solving specifically including those not modelled by the instructor, cannot be used to identify and characterize students in science, mathematics, engineering or related fields who are enrolled in mandatory first year math course at community colleges across US (Blair et al. 2006). Their fluency and flexibility with mathematical content is minimal yet that doesn't mean they are not creative. According to Prabhu (2015), *Creativity in teaching of remedial mathematics is teaching gifted students how to access their own giftedness*.

The lack of educational research on creativity with the general population has been pointed out by Sriraman et al (2011) "The role of creativity within mathematics education with students who do not consider themselves gifted is essentially non-existent" (p.120). Clearly if such research is to be done, what we (Prabhu and Czarnocha, 2014) shall refer to as the 'democratization' of creativity, one needs a definition in which to measure creativity that does not exclude ordinary students.

Before reviewing Koestler theoretical framework for creativity and its relationship to the raised issue it is useful to consider on why one might be interested in creativity with students who do not consider themselves gifted in mathematics. Liljedahl (2013) studies students he classified as 'resistant', "Many of the students would describe themselves as math phobic, math-incapable or a combination of the two. They usually have negative beliefs about their abilities to do mathematics, poor attitudes about the subject, and dread the thought of having to take a mathematics course"(p.256) exposing them to creative pedagogy and asking them to write about their experiences. He reported that they experienced illumination or "Aha" moments both while listening to the instructor and while working problems out for themselves the latter being much more common and that such experiences frequently did alter their attitudes towards mathematics. One of the basic insights of V. Prabhu is that students who struggle with mathematics especially in underserved populations frequently have a negative affect towards mathematics and perhaps the only way to reach such students is through a pedagogy that guides and exposes students to their own creative potential, through a social contract (Surrazy & Noventná, 2013) that encourages them to own their learning process what she would refer to as a Handshake and a Compromise.

Furthermore, the lack of such pedagogy propagates within a society in which teachers such as those described by Liljedahl (2013) (before his research project) will pass on their own negative view of mathematics to children who will follow their lead and then pass on such negative views to friends and children, in such a way an entire society itself begins to accept that is cannot do mathematics and that such a situation is a natural state.

KOESTLER THEORETICAL FRAMEWORK FOR CREATIVITY

Koestler (1964) describes the mechanism of creativity in terms of an (hidden) analogy between two or more previously unrelated frames of reference:

> "I have coined the term bisociation in order to make a distinction between the routine skills of thinking on a single plane as it were, and the creative act, which … always operates on more than one plane" (Koestler, 1964, p. 36).

The terms matrix and code are defined broadly and used by Koestler with a great amount of flexibility. He writes, "I use the term matrix to denote any ability, habit, or skill, any pattern of ordered behaviour governed by a code or fixed rules" (p. 38).

> The matrix is the pattern before you, representing the ensemble of permissible moves. The code which governs the matrix…is the fixed invariable factor in a skill or habit; the matrix its variable aspect. The two words do not refer to different entities; they refer to different aspects of the same activity. (Koestler, 1964, p. 40)

Thus, for Koestler (1964), *bisociation* represents a "spontaneous flash of insight ... which connects previously unconnected matrices of experience" (p. 45). That is the"…transfer of the train of thought from one matrix to another governed by a different logic or code …" (p. 95).

The focus of the Act of Creation Theory is on the bisociative leap of insight, that is, an *Aha!* moment, or a *moment of understanding*, — a phenomenon that contains both an affective component of the 'Aha' moment and a cognitive component of the synthesis of two previous unrelated matrices of thoughts, the hidden analogy as Koestler would refer to it. These components in so far as they can be observed amongst the general population suggests Koestler's framework as suitable for measuring and analyzing the creative aspect of self discovery during learning.

REFLECTIE ABSTRACTION AND BISOCATION

Bisociation can be observed within the moments of discovery in mathematics, which lead to self-discovery[7] during the many leaps of insight. They take place when listening to the instructor or during work as a full class, in a small group or individually, and while they do, it is useful to reflect upon the relationship between bisociation as the Koestler's framework for creativity with reflective abstraction used by Piaget to explain learning. Like the mechanisms of bisociation, it is based upon conscious reflection and abstraction of solution activity.. According to Piaget, construction of schema and cognitive change:

> …proceeds from the subject's actions and operations, according to two processes that are necessarily associated: (1) a projection onto a higher level (for example, of representations) of what is derived from the lower level (for example, an action system), and (2) reflection, which reconstructs and reorganizes, within a larger system, what is transferred by projection. (Piaget & Garcia, 1989, p. 2)

Dubinsky (1991) employs Piaget's mechanisms of reflective abstraction on actions and processes within the process/object duality of concept development. Dubinsky considers *interiorization*, *coordination*, *generalization*, *reversibility* and *encapsulation*. Interiorization involves the internalization of processes or actions, while reversibility involves reflection upon the inverse of a known process. Coordination of two processes is an integral part of reflective abstraction as defined above when after projection the learner reflects upon and begins to reconstruct and reorganize his original knowledge coordinating it with processes and structures available in the larger structure. The original projection of knowledge onto the higher plane and the reflection required during the coordination of this projected knowledge with processes of the higher plane generalizes and abstracts the learners knowledge to the higher structural level. This work of Dubinsky has laid the foundation for APOS or action-process-object-schema theory in which the learner transitions between through three types of knowledge or understanding the action level: actions, processes, and objects, which are themselves organized into schema. "An action is any physical or mental transformation of objects to obtain other objects. It occurs as a reaction to stimuli which the individual perceives as external "(Cottrill et al 1996,p.2), in contrast to a process in which the individual is in control and can reflect upon the transformation. Processes are the result of the Interiorization of actions and at the process level an individual can employ the reflective abstraction processes of coordination and reversibility. The object level is obtained through the final encapsulation processes.

Simon et al. (2004) ascertain that such reflective abstraction, or more specifically, coordination, is brought about by comparisons of the solution activity and its effect in relation to the expected outcome:

> Thus, within each subset of the records of experience (positive or negative results), the learners mental comparison of the records allows for recognition of patterns, that is, abstraction of the relationship between activity and effect. Because both the activity and the effect are embodiments of available conceptions, the abstracted activity-effect relationship involves a coordination of

[7] Self discovery refers to the affective dimension of learning mathematics during the creative process of giving personal meaning to new knowledge. With highly motivated students it would refer to the affective dimension of bisociation as they become highly motivated to learn math and think outside the box. With resistant students in "underservered" populations it refers to the affective dimension during the transition from habits and acceptance of failure to one of personal responsibility for one's learning process. One discovers ones potential and gains self confidence

> conceptions … Note the activity and the effect are conception-based mental activities, our interpretation of Piaget's notion of coordination of actions (Simon et al., 2004, pp. 319-320)

This interpretation of Piaget's mechanism of coordination broadens the scope of reflection upon actions to reflection upon solution activity. In Koestler's framework for problem-solving, the experiential mental records are matrices, and coordination occurs initially between the individual's record of possibly related matrices and the given problem situation. A straightforward analogy, or association, is characterized by an isomorphic mapping, or a projection, of a matrix to the current situation, resulting in assimilation. When no such equivalent matrix is found, the solver searches for a matrix that, although not previously considered completely analogous, is suddenly seen as giving structure and meaning to the present situation. The hidden analogy comes to light. When this realization is led by intuition, Koestler refers to it as bisociation, and likens it to the process often exercised by mathematicians and scientists to exploit the structures of one discipline to give meaning, through analogy, to a problem in another.

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SELF-CONTROL AND SELF-MONITORING PROCESSES OF GIFTED STUDENTS IN MATHEMATICAL PROBLEM SOLVING SITUATIONS

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Abstract. *Gifted students are thought different from average students in many ways. It is probably one of the most prominent features of gifted students is quality of their self-regulated processes. The purpose of this study was to investigate gifted students' self-control and self-monitoring processes, which are components of self-regulated learning. For this purpose, we interviewed with 3 gifted secondary students. We designed 10 problem solving sessions with using think aloud protocol. As a consequence of this study we saw that gifted students used a wide range of both self-control and self-monitoring behaviours in mathematical problem solving situations. They leaved or regulated these self-control behaviours when self-control behaviours did not work in solving problems. They also made decisions about self-control behaviours with self-monitoring process.*

Key words: Gifted students, self-regulated learning, problem solving

INTRODUCTION

Calculating an IQ score by using standardized tests and determining whether the person is gifted or not has been widely criticized due to the fact that they are not built upon cultural foundations (Colangelo & Davis, 2003). As the improvements in psychology and cognitive science and theories about how learning occurs increase, the term gifted is re-described and re-conceptualization. Sternberg (1986), one of these researchers, stated that classical intelligence determining tools such as IQ tests measures only a part of the intelligence. Again, Sternberg (1986) suggested describing intelligence as a person's growing abilities instead of a fixed characteristic of a person. In literature, there are some views which claim that while evaluating mental abilities with IQ scores, the qualities such as creativity, flexibility, internal motivation, self-regulation etc. must be also taken into consideration (Calero, García-Martín, Jiménez, Kazén, & Araque, 2007). On the other hand, many researchers have predicted that students who have strong self-regulated learning behaviours are more successful (Pintrich, 2000; Zimmerman, 2000).

Self-regulated learning

Self-regulated learning theory can be seen as a comprehensive structure according to the social cognitive perspective while attempting to figure out processes that plays a part in students being active agents in their learning process. This theory generally features the following three cyclical phases: forethought, performance, and self-reflection (Zimmerman, 2000). In the forethought phase, during which self-regulated learners are faced with tasks, they cognitively analyse these tasks by activating their prior content knowledge and metacognitive knowledge, setting goals, and making strategic plans. This phase also involves motivational components, including goal orientation, self-efficacy, interest / task value beliefs, and the perception of task difficulties. The performance phase includes selecting and adopting cognitive and motivational strategies in order to reach their goals (self-control). Self-regulated learners are aware of these strategies and monitor them during this phase (self-monitoring). As for the self-reflection phase, self-regulated learners make judgments of their progress according to a standard and ascribe causal attributions to their performance. These self-reflection processes influence their future forethought phases and, finally, the cycle is completed (Pintrich, 2000; Zimmerman, 2000).

In mathematics, self-regulated learning helps students to plan, guide, and monitor their thinking when they are faced with challenging problems. There are various studies related to the subcomponents of self-regulated learning in this area. While some of them examine students' strategy use (Montague, 1991; Pape & Wang, 2003), others investigate the relation between students' motivational beliefs and problem solving (Pajares & Miller, 1994). Most of these studies are conducted in an experimental context. Also, there are many studies in problem solving which focus on metacognitive activities and emphasize strategic behaviours that are important parts of self-regulated learning (Artz & Armour-Thomas, 1992; Cifarelli, Goodson-Espy & Chae, 2010). Studies that examine gifted students in the mathematical problem solving literature are usually designed by comparing these students with average students (Montague, 1991). Despite research over the last thirty years that indicates that self-regulated learning supports academic learning, gifted students' self-regulative behaviours have not been sufficiently studied (Dresel & Haugwitz, 2006). There is an increasing demand for studies that can explain and reveal gifted students' self-regulative behaviours in academic settings in order to better understand gifted students' learning processes. The aim of this study is to investigate gifted students self-control and self-monitoring processes while solving mathematical problems.

METHODOLOGY

The Participants

In Turkey, gifted students are usually in the same classes with normal students and receive the same mathematical education program with other students in classrooms. However, some private schools might organize after school or weekend courses which contain extracurricular activities for the students who display high performance. The content of these courses mostly include mathematical subjects that are at a more advanced level than the average level of the class. Besides that, students are placed in Science high schools and Anatolian high schools in which the students with higher SBS (placement test for enrolling in high school) scores are placed. Based on their SBS score they receive a more extensive education (Sak, 2011). In addition to that, ever since 1993, gifted students has been receiving education in their out of school time in Science and Art Centres which are affiliated with the Ministry of National Education. Three gifted secondary students of a Science and Art Centre which is located in a big city in Turkey have participated in this study. Nicknames were used while presenting the research data.

Data Collecting Process and Tools

In order to find out which self-regulated behaviours the students use while solving mathematical problems, 10 problem solving sessions were conducted. The students were asked to solve the problems through think aloud protocol. It took approximately 20 minutes for the participants to solve the problem in these sessions. Since it was observed that in problem solving sessions the participants were unable to transfer many of their explanations onto the paper and vocally express some of the methods and strategies they used; the participants were asked to explain how they solved the problem and the operations/drawings, which were not clear in their solving papers.

The Data Analysis

The qualitative data analysis, designated by Auerbach and Silverstein (2003), was adopted while analysing the data collected throughout the research. Before starting the data analysis, a coding protocol was created based on the self-regulation theory. This approach was chosen so as to elaborate the existing self-regulatory behaviours within the context of problem solving. Constant comparative analysis, one of the grounded theory methods, was also used during data analysis (Glaser & Strauss, 1967). Moreover, cross case analysis was used to identify similarities and differences between each case, in other words each participant (Yin, 1994). In the findings, only this cross case analysis will be examined.

FINDINGS

The self-regulation behaviours displayed by the students during the performance phase are examined in this section according to two main processes which are self-control and self-monitoring. In addition to that, their purposes in their behaviours were also taken into consideration. Self-regulation behaviours were analysed with the aim of understanding the problem, obtaining the solution to the problem and questioning the answer. Some examples can be seen in Table 1. We will give some examples which are not stated in Table 1 for two main processes according to the problem (Posamentier & Salkind, 1988) below:

The measure of a line segment [PC], perpendicular to hypotenuse [AC] of right ΔABC, is equal to the measure of leg [BC]. Show [BP] may be perpendicular or parallel to the bisector of <A.

Self-control

In order to understand the problem, gifted students *used visual models based on the given information* and also *read the problem by dividing it into pieces* as self-control behaviours. While try to obtain the solution, students used various self-control behaviours. Demir *used visual models* in order to describe the solutions he planned in his mind about the information given in the problem. Similarly, Ahmet *used symbols* in order to find out what is asked in the problem Ege tried to *associate with the other mathematical subjects* since he thought it would be helpful for solving the problem. Another self-control behaviour used by Demir and Ege when they could not find the solution is *working backwards*. Ege assumed that it met the requirement that “[BP] should be parallel to bisector of A angle”, which was given the problem, and; tried to solve this problem. Ege, who were unable to obtain a result by using that way, tried to solve the problem by *examining specific case,* in other words; by assuming that there is a double relation among the measures of acute angles of the triangle.

Self-monitoring

Ahmet demonstrated one of self-monitoring behaviours towards understanding the problem. He realized that he was unable to understand what [*PC*] meant in the problem; and said that ''so, from where do I take the CP? From right or left?''; and then *thought over the requirements given in the problem*:

I am trying to figure out how I indicate that it is right or parallel now. [after 2-3 seconds] Is it right or parallel? Because we draw two BP's. [thinks for 5 seconds] Well, since here it says right or parallel, we assume that it is either right or parallel according to BP straight line but why does not it tell us that it is parallel?

Process	Examples of behaviours from 10 problem solving sessions
<u>Self-control</u> Understanding the problem	*Going back and reading the problem again* in order to make sure what is asked in the problem *Paraphrasing the problem by using their own words* in order to understand the problem better
<u>Self-control</u> Obtaining the solution to the problem	*Reducing method* when they firstly considered the smaller numbers instead of the numbers standing for the problem *Finding a pattern* by examining whether there is an order among the numbers which are smaller than the numbers in the problem. *Making a generalization* by considering the same for the number in the problem when they managed to find a pattern *Examining every possible case* in order to meet the condition in some problems
<u>Self-monitoring</u> Understanding the problem	*Thinking over whether they were moving in accordance with the requirements* in order to correctly process and/or confirm whether they accurately transferred; to overcome any trouble they encounter about the information given in the problem *Reading the problem again* when they were unable to understand
<u>Self-monitoring</u> Obtaining the solution to the problem	*Thinking over their solution* in order to direct themselves for the solutions to the problems; to decide whether these solutions are practical; to see if the self-control behaviors they used while solving the problem worked or not *Turning back and reading the problem again* when they doubted that there was a requirement which they did not notice *Thinking over the solution with their previous knowledge* in order to come over a situation which they got stuck in *Cleaning his mind / Changing his position / Scratching his hair* in order to increase his concentration
<u>Self-monitoring</u> Questioning the answer	*Trying to justify the solution through a second method* *Turning back to the problem and read it again*

Table 1: Gifted students' self-regulative behaviours while solving the problems

It is presumed that Ahmet displayed this behaviour in order to figure out how to represent the requirements given in the problem on the visual model he created.

Gifted students display self-monitoring behaviours towards obtaining the solution to the problem. For example, Demir and Ege *asked questions and talked to themselves* in order to produce ideas for the solution, not to make any operational mistakes and to regularly produce ideas when the solution does not work. Demir gave the following statement after indicating that "[BP] is parallel to the bisector of A angle" in the problem:

But how do I [stops for 1 second] indicate that it is right? [stops for 1 second] I have no idea. That it is right. I mean BP to A angle and A angle's [goes over PC] [thinks for 5 seconds] how come BP is right to A angle? [thinks for 3 seconds] To bisector of A angle? [thinks for 5 seconds] I have no idea about that but it seems to me I had indicated that it should be parallel [thinks for 3 seconds, playing with his hair] but how consistent is trial and error method? [thinks for 7 seconds, playing with his hair] That's it. I cannot indicate that it is right. How could it be right after all, based on the given information?

Here it is seen that Demir *talks to himself and asks himself questions* about how to satisfy the requirement of "[BP] being right to bisector of A angle". It was seen in the interview that Demir was aware that this self-monitoring behaviour he displayed helped him produce ideas for the solution. For questioning the answer of this problem as one of the self-monitoring behaviour, it was observed that Ege *revised their solution* after finding out the answer in order to make sure if he correctly solved it or not.

CONCLUSIONS

Gifted students revised the self-control behaviours towards understanding, through self-monitoring behaviours and regulated these self-control behaviours again when necessary. After understanding the problem, the gifted students displayed self-control behaviours towards obtaining the solution to the problem. They either quit or accordingly re-regulated the self-control behaviours which did not take them to the solution or help them proceed while solving the problem. Krutetskii (1976) stated that in mathematics, one of the qualities of gifted students differing them from the other students is the *flexibility* they displayed while solving problem.

It was concluded that the gifted students decide on the most appropriate one through using self-monitoring behaviours. In the interviews, it was observed that the students regularly and consciously monitored whether the self-control behaviours helped them find the solution or not. Also, it was seen that the students consciously checked and regulated cognitive processes−which is an indication of a good problem solvers (Lester 1994). By taking into account of this, it was concluded that the educational activities involving the gifted students should be designated in a such way that it will enable the students to realize their own potential, to cognitively and metacognitively develop and to motivate themselves (Diezmann & Watters, 2002). Furthermore, it is also convenient to face these students with challenging mathematical problems or activities.

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ORAL PRESENTATIONS 2.1.

Creativity in early years

Chair of the session: **Alex Friedlander**

COMBINATIONAL PLAY BETWEEN MATHEMATICAL DOMAINS AS ONE DIMENSION OF MATHEMATICAL CREATIVITY IN THE EARLY YEARS

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Abstract. *A child's growth in mathematics involves more than just mastering mathematical problems with given algorithms. Children have to observe their environment with open eyes so that they are able to pose mathematical questions and to work creatively on mathematical problems. Although there is a common sense about the importance of creativity in the development of mathematical thinking in the early years, it seems hard to define what mathematically creative processes and abilities look like. Therefore the paper introduces the combinational play between mathematical domains as a mathematically creative ability of preschool-age children. Data are collected in the interdisciplinary project MaKreKi (mathematical creativity of children) in which researchers of mathematic education and psychoanalysis investigate the development of mathematical creativity.*

Key words: mathematical creativity, mathematics in the early years

COMBINATIONS AS ONE DIMENSION OF MATHEMATICAL CREATIVITY IN EARLY YEARS

Definitions of mathematical creativity differ with respect to several assumptions. In many approaches, creativity is seen as the individual ability of a person in the sense of divergent thinking (Guilford, 1967), the abilities to produce fluent, flexible, novel and elaborated solutions to a given problem (Torrance, 1974) or the ability to produce unexpected and original work that is adaptive (Sternberg & Lubart, 2000). In another research tradition creativity is seen as embedded in a social process (e.g. Csikszentmihalyi, 1997; Sriraman, 2004; Vygotsky, 2004), in which creativity is not solely located in a person's cognition, but is also accomplished in the social interaction among members of society. So, it is important to distinguish between the social, creative *process* in which more persons are participating and the individual, creative *ability* that emerges during this process of social interaction. The paper focuses on the reconstructed individual mathematically creative activities of young children of kindergarten age who are participating in mathematical situations of play and exploration (Vogel, 2013). This span of life commonly is seldom taken into account in studies about mathematical creativity, but there is an enormous literature about young children's creativity in general (e.g. Craft, 2013; Saracho, 2012; Vygotsky, 2004). The research question stresses these findings by asking how mathematically creative activities of children at preschool age can be identified and described. At the onset of the paper, some theoretical assumptions about creativity are presented. For the analysis of mathematically creative activities of children, which follows, the paper focus on "combinational play" with reference to Finke (1990, p. 3) as one aspect of creativity in the early years.

Following Vygotsky's approach, creativity depends on the person's previous experiences. For Vygotsky, the central ability of creativity is imagination that occurs in play situations. There is no opposition between imagination and reality, because imagination is based on elements taken from reality (Vygotsky, 2004, pp. 14-15):

"…the creative activity of the imagination depends directly on the richness and variety of a person's previous experience because this experience provides the material from which the products of fantasy are constructed."

Imagination describes a circle. It takes fragments of reality and transforms them and as these new fragments take shape they reenter reality. For example, often a child's play is an echo of what adults do, but it is not simply a reproduction of what it has experienced, but rather a creative reworking of the impressions it has acquired. It combines them and uses them to construct a new reality, one that conforms to its own needs and desires (ibid). Also Finke (1990) describes the ability of doing combinations ("combinational play", p. 3) as one dimension of creativity. He states that "real creativity comes from using the things we create, not creating the things we use" (ibid, p. 3). This aspect of combination is also emphasized in research about the mathematical creativity of adult mathematicians (e.g. Ervynck, 1991; Poincaré, 1948; Sriraman, 2004): Ervynck (1991) describes mathematical creativity as the ability to create a new and *useful* mathematical concept through combining mathematical concepts or relations. This definition is similar to Poincaré's ones, who sees the ability to choose from the huge number of possible combinations of mathematical propositions a *minimal collection* that leads to the proof as one fundamental aspect of mathematical creativity (Poincaré, 1948).

Regarding the age group relating to interest, one cannot assume that young children who are at the beginning of their development of mathematical thinking would work like mathematical professions and choose only *useful* and *minimal collections of proofs*. It is more likely that their perspective on mathematical situations differs from the adult ones (Münz, 2014a; Münz, 2014b) and that their combinations may be of another kind. Therefore, one has to distinguish between the mathematical creativity of professional level and of school level (Leikin, 2009; Sriraman, 2004). For Sriraman (2004) one aspect of mathematical creativity in school children is: "the formulation of new questions and/or new possibilities that allow an old problem to be regarded from a new angle" (ibid, p. 120). The aspect of combination lies on adding other perspectives to a given known problem. Probably it is not possible to capture the concept of creativity in one definition. But the aspect of combination in the sense of *combinational play* can be considered as one aspect of a mathematical creative ability, which may be observed in adults as well as in young children while dealing with mathematics. Therefore three cases are presented in which young children change the perspective of a mathematical situation by combining different mathematical domains. First some brief information about the empirical and methodology approach is provided.

EMPIRICAL APPROACH AND METHODOLOGY

The sample of MaKreKi is based on the original samples of two projects that are conducted in the "Center for **I**ndividual **De**velopment and **A**daptive Education of Children at Risk" (IDeA) in Frankfurt, Germany. One of the projects is a study of the evaluation of two prevention programs with high-risk children in day care centers (EVA). It examines approximately 280 children. The second project is a study of early steps in mathematics learning (erStMaL). This project includes approximately 150 children. Thus the original sample contains 430 children. We asked the nursery teachers of the two original samples, whether they knew children in their groups who show divergent and unusually sophisticated strategies while working on mathematical tasks. We could identify 37 children, who seem to work creatively on mathematical problems.

For the examination of the development of mathematical creativity in the selected children, we introduced mathematical situations of play and exploration, which are described in didactical design patterns (Vogel, 2013). The conceptual origin of each situation lies in one of the five mathematical domains: Numbers & Operations, Geometry & Spatial Thinking, Measurement, Pattern & Algebraic Thinking, Data & Probability (Sarama & Clements, 2008). They are designed in a way that the children can demonstrate their mathematical potential in the interactive exchange with the other participants. An assisting adult is supposed to present the material by sparingly giving verbal and gestural instructions. With respect to the longitudinal perspective, the children are observed every six months while they work on two mathematical situations of play and exploration. All these events are video taped with two cameras.

Due to the problem of defining mathematically creative processes in early childhood I have focused of combination as on aspect of mathematical creativity. All videotapes are coded concerning this aspect and I could identify mathematical (sometimes surprising and unusual) combinations nearly in all tapes. For the following three analyses I have chosen explicit combinations of children in the sense of *combinational play* by combining different mathematical domains. These sequences are analyzed by interaction and argumentation-analysis, based on the interactional theory of learning mathematics (Brandt & Krummheuer, 2001). They focus on the reconstruction of meaning and the structuring of the interaction process. The applied analysis of interaction is derived from the ethnomethodologically based conversation analysis, in which among others it is stated that the partners co-constitute the rationality of their action in the interaction in an everyday situation, while the partners are trying constantly to indicate the rationality of their actions and to produce a relevant consensus together.

EMPIRICAL CASES

Sina in the “Wooden animals – Situation”

Sina (S, 5;4 years old), Victoria (V, 5;8), Konrad (K, 5;7) Bahar (B, 4;7) and an assisting adult (A) are dealing with little wooden animals like dogs, cats, chicken, etc. varying in colors and numbers. The children have to find out how many dogs or chicken there are and which animals are the most. So this mathematical situation of play and exploration originates in the domain of Numbers and Operations. First the group decides to sort the animals by faunal species (see Figure 1). Afterwards they begin to count the blue chicken and realize that there are 35 blue chickens and one black one, which they classify as a little chick. Because of this high number the counting process is very long. Then the assisting adult asks whether they have more blue or more green dogs.

Figure 1: Wooden animals sorted by species

Figure 2: René's and A's snail shell

Figure 3: Rows of ladybugs

1 S: more green ones

2 K: more green
3 A: how can we find out which one are the most/
4 S: counting
5 A counting/ do you have another idea/
6 K: calculating
7 A: calculating/ look I propose we put the dogs very close to each other so we can find out something/
8 S: oh yes then we put at one side the blue dogs against each other and on the other side the green ones and if the green line is longer there are more green dogs and if the blue line is longer there are more blue dogs
9 A: that's a good idea let's do this

Sina changes the arithmetical framing of the situation to a more measuring perspective. The placement in lines of the dogs allows an easy comparison of the length of the two lines. Therefore she has to combine two mathematical domains: Numbers & Operations and Measurement.

René in the "Rope – Situation"

In this mathematical situation of play and exploration, which has it's origin in the mathematical domain of measurement, René (R, 5;7), Marie (M, 5;6), Levent (L, 5;6), Chris (C, 5;0) and a member of the research team (A) compare ropes which have different colors (blue, green, red and yellow) and sizes (long, medium and short). The children and the member of the research team compare the rope sizes by holding two ropes close to each other. In this way they can determine if the two ropes are of equal length or if one rope is longer than the other. This kind of measurement is easy if they use ropes of a small size but it seems to be clumsy if they use long ropes, because the medium and the large ropes are longer than the arm-span of the children and the member of the research team.

The group decides to arrange the ropes according to size. This "structuring phase" ends up by putting all ropes together, so the ropes are not sorted yet. Than René comes up with a surprising idea:

1 R: but I can do something like this from these ropes\ puts the yellow rope, which he holds in his hands on the floor
2 A: what can you do/ show it to me
3 R: pushes with his feet a long red rope towards Marie a snail shell gets on his knees and puts the rope on the floor in the shape of a spiral
4 A: a snail shell\ looks towards Marie look can you do this too/

After shaping some snail shells the assisting adult asks if his green one is bigger. René points at the green snail shell and says this one is smaller (see Figure 2). The original problem comparing the ropes regarding their length is now being observed from another (possibly geometrical) perspective. The other members of the group adopt René's idea and so everyone makes an Archimedean spiral, which René has called snail shell. The spiral form of the ropes now allows a direct comparison, which seems to be for example simpler and more comfortable than holding two long ropes together. René combines the two mathematical domains Geometry & Spatial Thinking and Measurement within his proposal of shaping spirals.

Marie in the "Ladybug– Situation"

Marie (M, 4;9), René (R, 4;10) and assisting adult (A) arrange cards of ladybugs which differ in size, color and shape of their marks. René ascertains that the red ladybugs are the most and Marie thinks that there are also many green ones and suggests comparing their quantities. René tries to count all the red ladybugs and Marie tries to help, but because of the high number of ladybug cards, which is higher than the children's number concept, they have to find another way to compare the quantity of the red and green bugs.

1 M: so which ones are the most/
2 A: hmm how can one find out/
3 M: with what … and what reaches to the end of the table is the longest ones
4 do we want to look if this idea works/
5 A: what? say it again/
6 M: let's make a big row until over there points with her finger at the end of the table
7 A: mhm
8 M: and the row that reaches to the edge of the table is the longest one

Marie suggests that the bugs be placed in rows sorted by color (see Figure 3), because a comparison of the lengths of the rows allows the group to determine which one has the largest number of ladybug cards. In her proposal, Marie associates the domain of Numbers & Operations with the domain of Measurement.

SUMMARY

As the three examples of Sina, René and Marie demonstrate, the ability to change the perspective of a mathematical problem or question by combining two different mathematical domains can be seen as one expression of mathematical creativity (most likely not only) in the early years. In all cases this *combinational play* can be regarded as useful to solve the given mathematical problems. In contrast to Sina and Marie, who work very closely on the given task, René shows a relatively loose working attitude. His idea of shaping the ropes may not first be developed to solve the problems of size comparisons, but the group uses it to solve them. So mathematically creative abilities in the early years occur by solving problems and creating things as well as by implementing ideas and later using this creation (e.g. to solve a problem). Therefore the children need some kind of free space in which they are able to realize their mathematical *combinational play between mathematical domains* even though it may, in the beginning, stray away a little from the original problem.

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LOOKING FOR THE WAY TO SUPPORT CHILDREN'S MATHEMATICAL CREATIVITY

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Abstract. *In Prague, students – future primary teachers - from Faculty of Education, during 4 years of their studies, have a chance to gather experience on how to work with young children to support their interest in mathematics. In this paper, we describe some observations gathered during "the afternoon class" with 7-8 year old children while solving non-typical tasks. We juxtapose a child's actions with a teacher's attitude. It shows that the teacher does not always recognize how to support children's mathematical creativity.*

Key words: joy from finding solution, non-typical task, persistence, supporting creativity, teacher's attitude.

ARE CHIDREN GIFTEN IN MATHEMATICS? (INSTEAD OF INTRODUCTION)

We will start with some observations gathered by mums observing their little children:

Tom, 15 months old. Tom loves to play with balls. He calls the ball ‚bał'. While he was eating a salad with peas, he pointed at them with a finger and called loudly, "bał , bał". Similarly, when he saw my earrings in the shape of a sphere, Tom touched them with a finger and cried, "bał , bał". Likewise, when the object in his booklets is viewed as a circle he is doing the same.

Gabryś, 2 years old. One of Gabi's toys is a box of stamps in different shapes. Gabryś removed all the stamps from the box and then bounced stamps on a special board. Initially, he does stamps in just any place, after a moment of a reflected thought he erases all characters and starts to bounce stamps symmetrically in the corners.

Maja, 2 years and 3 months old. Two sisters got 4 jellies. Mum said: please share them. The girl answered: two for Susan, and two for me. For mum there aren't any.

Martinka, 2 years and 6 months old. Martinka with her mom returned from a visit at her grandma's. During the journey, they took a bus, the subway and then a bus again. When Martinka arrived home, she joyfully ran to her dad's closet, where he had flat shape boxes. She took five boxes, deployed them in roughly the same distance in the corridor and said, "Mom, come on, let's play the subway. We will embark on other stops and get on again". The following week, she "relocated" the game to the garden, where the bus was a garden wheelbarrow, the driver was their grandfather and she was the passenger, who was picked up at bus stops in the form of trees. For another week, at home, she made a bus of Lego-blocks, the stops were the things that were currently lying on the carpet, the passengers were small figurines and Martinka at the final stop was asking the question: how many passengers disembark?

In each of the examples it can be seen that even in the case of small children, pupils are highly sensitive to mathematical phenomena. They are sensitive to shapes, regularity in arrangements of objects, mathematical operations and to the change of the number. It is also seen that matemical thinking is quite natural for children. This is not an artificial world, quite the contrary - it is the world in which they live.

Perhaps similar examples have led A. W. Krutieckij (1968) to the conclusion that the innate mathematical giftedness can be seen in children's behavior. He also claims that when these abilities are properly developed, they will take the form described in his model of mathematical giftedness. In spite of many research on mathematical giftedness, there still is a lack of knowledge about how to grow natural children's ability in this domain. The range of children's experiences who start schooling causes that, right from the beginning, school's offer may be lower than it should be. On the other side, it is also difficult to judge definitively that the child has no mathematical inclination, perhaps because it did not have the opportunity to present it yet. Additionally, many researchers stress that the development of mathematical abilities should be done in two ways - by supporting the mathematical competence and adequate development of the personality (Brandl, 2011), but these two strands are not uniformly supported by teachers.

In the years 2007 – 2010 E. Gruszczyk-Kolczyńska (2011) has conducted the research on a group of 182 preschoolers and younger school classes students, to check their math abilities. Her research was inspired by the Krutietskij theory. E. Gruszczyk-Kolczyńska wanted to see how the characteristics of mathematical talent, highlighted by Krutietski, relate to children in the preschool and early school education level. The research tool was a set of 13 packages related to the specific areas of school mathematics. She decided that the mathematical abilities of the child are proved by the fact, that in at least one area of mathematical activity he/she shows high competence. The results of her research show that about two thirds of the children met this criterion. This means that the vast majority of older preschoolers and young students are gifted with mathematical abilities. In this group there are also highly gifted children.

Unfortunately, teachers are not able to recognize the competence of their pupils, which is a further result of the research conducted by E. Gruszczyk-Kolczyńska. During the first year in school children tend to lose the joy of solving problems, cease to be creative and critical, are passive and accept adult's opinions.

THE WAY TO SUPPORT CREATIVITY

Milan Hejny's methods of teaching mathematics

Nowadays in Czech Republic M. Hejný method exists, which is based on 12 principles (www.h-mat.cz), and many of them support children's creativity. We would like to highlight some of them.

4. Character development: The children learn to know themselves what is right for them; they respect each other, know how to make decisions, and accept the consequences of their actions.

5. True motivation: all mathematical problems are designed so that children enjoy solving them. The right kind of motivation is the internal kind, not forced by any outside factor. Children find the solutions to the problems thanks to their own efforts.

7. Enjoying mathematics: the most effective motivation derives from a child's feeling of success, from their genuine joy of having solved an appropriately demanding task. Children who have this motivation do not experience the "mathematics paralysis" that has become legendary in traditional education.

Amounts of areas of high competences	Age		
	4 year old n = 41	5 year old n = 40	6 year old n = 59
1	17	6	11
2	7	2	7
3		4	3
5 and more	2	8	14
5 and more, in %	5%	20%	19%

Table 1. Manifesting mathematical abilities in different age groups

8. Personal knowledge: pupils discover mathematics themselves. They collects a range of experiences, which they talk about. They each explain their own theories, and subsequently test them on further problems. Throughout the process, they understand what they are doing.

This method is designed not only for gifted students – on the contrary, M. Hejný believes that it is a method for „teaching everybody mathematics". We support this statement, and we strongly believe that such way of work is the only one, even for gifted children. But work with 'gifted children' is challenging for teachers. Usually, they are very strong as personalities, they have a high self-esteem, they like to think independently. How to be prepared to work with gifted children? How to asses if a child is gifted or not in class?

Afternoon classes for young mathematicians

In Prague, students – future primary teachers - from Faculty of Education, during four years of their studies, have a chance to gather experience on how to work with young children to support their interest in mathematics. They organize afternoon classes for children who want to be envolved in any mathematical activities. Children are on various levels of giftedness (weak, average, excellent). There are five groups (each has approximately 15 pupils), one group of grade 2 (7-8 years old), two groups of grade 3 (8-9 years old), two groups of grade 4 (9-10 years old). They work one hour per week. Each group is led by about 3 students. One week before classes, they choose topic of their next work with pupils, then they discuss a method and aims of work. They write their scenarios to a supervisor (this is one of the authors) who checks it mainly from the mathematical point of view. After each afternoon classes, students reflect on what has happened during these classes. They learn through their own experiences how to improve their methods of work with pupils and not to limit children's creativity.

There are two aims:

1. To offer children some activities to show them a 'different face' of mathematics – show them untypical tasks, enable them to use an open approach for finding solutions and so on.

2. To leave students to discover effective methods of work with pupils. In this paper we describe one of the activities prepared by a university student and the observation which was done after this afternoon class.

A BRIEF LOOK AT ONE OF AFTERNOON CLASSES

Scenario

The task was solved in the framework of one afternoon class, in November 2014. There were 17 children, aged 7-8. They have solved the following problem (Baggett, Erhenfeucht, 1998):

7 squirrels decided to have a race. There were 31 peanuts as prizes. An owl decided that the faster one should get more than the slower one. Each squirrel should get a different number of peanuts. How to do it? How many solutions are there?

This task is not typical. First of all, there are several solutions for this task. It could be solved by using a variety of strategies and methods, as well as many different forms of representation.

Student Daniela, leading the class, was prepared very carefully. She brought materials (nuts, beans, worksheets for kids). Before presenting the task, she had planned some preparations directed towards the expected problems in its understanding (like: what it means to put the numbers from the smallest one to the greatest, what it means to 'split among themselves'). Children have solved these problems in a dynamic way, manipulating nuts or positioning themselves in a row, acting as 'the numbers'. Daniela expected that children will work at different levels, and therefore she created a series of further questions, with gradual difficulties. She wanted to use them as hints for children. The supervisor suggested to abandon the hints and give children complete freedom to work on tasks. For this reason, Daniela was faced with the need to improvise a part of the course and spontaneously respond to what is happening in the classroom, regardless of the scenario.

Work on the task - case of Magda

Children worked in groups of two (three). They had paper, pens and beans that can be used for counting at their disposal. Magda decided to work independently.

First, she drew boxes and marked them from 1 to 7. Above them, she put the numbers from 10 in descending order. Before finishing creating a record, she checked how much there is still left. She did it by using beans (fig.1). She realized that had already exceeded the value of 31 so she didn't write the value above 7 and vigorously deleted the entire record (fig.2).

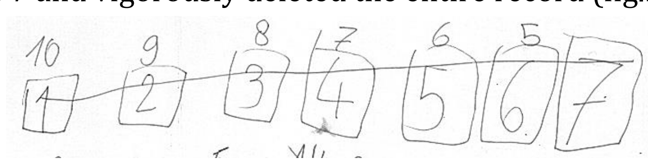

fig.1 fig.2

Underneath, she drew boxes numbered from 1 to 7 again. She tried to solve it again, she wrote 10 - 9 - 8 – 6, unless converted, that was too much. She deleted 6, wrote only 1, probably because she saw that there were other fourth places and there the nuts had to be given as well. Furthermore she wrote 0, because she knew that after 1 there has to be a smaller number, i.e. the number 0. This number (1) created a conflict situation: it is crossed out, then re-written and then crossed out again. Magda was in conflict: she wanted to give each squirrel any number of nuts and she had only 4 nuts and 4 places left, therefore each of them could have only 1 nut. But it was impossible because the fourth squirrel had to have more than the fifth. Once again, she returned to the whole process, cut all previously mentioned numbers of nuts. This time she did not draw 'numbered boxes' for winners, she focused only on the number of nuts and on writing numbers again.

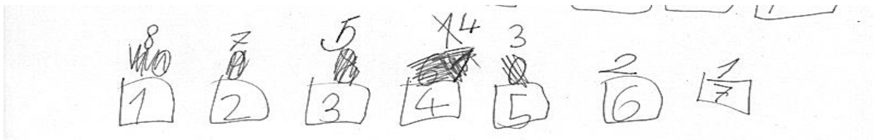

Fig. 3

The last two numbers are inclined, which shows that she began writing in a different order. She started from 1, than went from right to left. The shape 5 is strange - like writing it was combined with great deliberation. She probably knew that she consumed only few nuts for squirrels, and she still has a lot left. Therefore, she decided to use larger numbers and thus writes 7 and 8. As it can be seen, in this way Magda splitted only 30 nuts.

Daniela observed children's work. She discovered that they have some preliminary trials represented by different ways, however none of them had the right solution. Supervisor suggested Daniela to make 'a drama'. Seven children played a role of squirrels participating in the race and Magda was an owl, assigning rewards.

The girl treated it as another attempt to solve the task. She started from the first squirrel and gave her seven nuts. Then, with great deliberation, distributed the following awards - the second child got 6 nuts. Then, she recalculated all remaining nuts and, after a long moment of thought, she decided that the third child will have 4 nuts. Breaks and moments of reflection suggested that she still performs calculations in her head. After the allocation of the next three nuts for the fourth squirrel, she again thought for a long time - until the girl assesed that her initial estimates were not effective, however, she continued to work consistently. She assigned a decreasing number of nuts to other children. The need to preserve a dwindling number of nuts for prizes meant that the last squirrel did not get anything, but the prize pool has not been exhausted (fig.4). Then, the other children spontaneously gave the hints for continuation - each one should receive one more nut, starting from the first squirrel. As a result of such action also the last squirrel received the award (1 nut) and 30 nuts were distributed. In the meantime, 1 nut „disappeared". But children were not aware of it. So, for the proper solution it was enough to enlarge the prize of the first squirrel. Finding solutions was accompanied by a great euphoria (fig.5).

Daniela's reflection

When I think of the problem of squirrels, I think that this is a very challenging task for children. Therefore I spent a lot of time thinking how to find the easier forms of this problem. I was very worried that some of children will not know what to do, and this I didn't want to have. I invented a cascade of problems with different degrees of difficulties. My supervisor and her colleague came to me half an hour before the afternoon classes and completely changed my plans. They did not want me to give the cascade problems but to let children solve problems in their own way and then to discuss their approaches with them. During the classes, I felt that the work does not go forward. I didn't find even one child there to whom the task was clear. It was chaos. I felt relief when I got a hint to do 'a dramatization'. I did not expect that it could help the children find a solution. But so it happened, the children found the solution and they were very happy about it. Even so, after the classes, I felt dissatisfaction with myself. I was surprised when my supervisor, together with a colleague from Poland, came to me and thanked me for the fact that I gave space for the children and that the whole activities were so nicely conducted.

Fig. 4 Fig. 5

CONCLUSION

Girl Magda showed many attitudes of a mathematically gifted child. From the beginning of work she was very consistent in finding a solution. Repeatedly checked the results, failures did not discourage her. She tried to keep all conditions - even when seemed to be contradictory for her. Finding solutions was for her a very positive experience.

The example of these activities shows that working with children on the development of their creativity is extremely difficult and requires a lot of thought-out decisions. The key is to be able to work in the children's zone of possibilities - that the proposed action does not block them, but could create an open space for activity. It is necessary to observe children's capabilities and not expect immediate, perfect solutions. During afternoon classes, children worked in a relaxed atmosphere. They were not evaluated due to the solution, the only thing Daniela ('teacher') pointed out, was the need to maintain all the problem conditions. For her such a way of work was not easy, as confirmed in her statements. In her mind there was still the thought that in the classroom order must be kept and results must be seen. This project convinces us that there is a great need of work with teachers aimed at developing children's mathematical creativity. They (not only future teachers) are afraid of spontaneous, unplanned actions. They believe that the work in accordance with a well-planned scenario, guiding the pupil through a cascade of problems with different degrees of difficulties is a guarantee of success. The lack of an immediate idea, failure in having a quick solution often is assessed as an educational defeat. They often do not appreciate the effort of multiple attempts for solutions, do not see that a faulty solution leads children to discover a correct one. Analyzing with students situations such as described in this article can be a good way to build a proper attitude of teachers.

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IDENTIFYING MATHEMATICALLY GIFTED PRESCHOOLERS

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Abstract. *Considering the importance of early identification of giftedness, the present study purports to develop a set of identification tools for mathematically gifted preschoolers. The proposed set of instruments is based on the validated model by Pitta-Pantazi, Christou, Kontoyianni and Kattou (2011), according to which mathematical giftedness consists of three components: mathematical abilities, mathematical creativity and natural abilities. Finally, we propose suggestions for practical implications and further research.*

Key words: preschool, mathematical giftedness

INTRODUCTION

Identifying mathematical giftedness is a challenging process (Hoeflinger, 1998). Early identification is the first and pivotal step towards the improvement of the educational, social and emotional development of gifted children (Pfeiffer, 2002) and can maximize their contribution to society (Silverman, 2013). Despite the recognition of the value of early identification by the research community, only few studies have been devoted to developing mathematical giftedness identification tools appropriate for preschoolers.

What is notable is that studies conducted so far tend to assess mathematical giftedness by focusing solely on mathematical intelligence. However, such unidimensional and traditional tools cannot provide deep insights into young children's abilities (Kuo, Maker, Su, & Hu, 2010). This is one of the reasons why gifted preschoolers constitute one of the most 'underserved' groups in education (Shaklee & Handsford, 1992, p. 36).

Therefore, the field still lacks a set of identification tools that relies on a contemporary and multidimensional model of mathematical giftedness and is especially designed for children of early ages. This research gap along with the importance of early identification of giftedness substantiates the necessity of the present study. Applying the multiple-criteria approach, the present study intends to develop a range of tools for the identification of mathematically gifted preschoolers. Since there is no clear age definition provided in the literature for the term 'young gifted', in this paper it is defined as children 4-6 years of age.

LITERATURE REVIEW

The identification of mathematical giftedness in early years

Thus far, studies carried out with students of various ages assess mathematical giftedness though subtests taken from IQ tests. These subtests focus on the visual perception, spatial ability and pattern identification. Regarding mathematical giftedness in early years, the preceding studies have used only arithmetic subtests to detect gifted children (see for instance, Robinson, Abbott, Berninger & Busse, 1996).

Nevertheless, these traditional tools are not suitable for assessing mathematically gifted preschoolers. Primarily, such tests ignore the multidimensional nature of mathematical giftedness (Reis & Renzulli, 2011) and the diversity of children's cognitive traits (Hoeflinger, 1998). According to Kuo et al. (2010), young children's achievement in such tests is negatively influenced by their inability to focus their attention for the required time or to control their emotions. Finally, such tests require high neuro-muscular development which preschoolers do not possess.

A number of principles have been proposed by a number of researchers for the development of assessment tools for general giftedness (Coleman, 2003). These principles could be adopted for mathematical giftedness in young ages as well. The first principle pertains to the multiple criteria approach (Coleman, 2003); the use of valid, reliable (Pfeiffer, 2002) and non-traditional tools, such as creativity tests, teachers' and parents' nominations (VanTassel-Baska, 2005). Apart from these, the tools for giftedness identification should have a domain-specific character and allow the detection of children's potential (Lohman, 2009). Finally, they should minimize the likelihood of under-representing minority groups (Coleman, 2003).

Theoretical model of mathematical giftedness

The theoretical framework of the paper is based on the validated model of mathematical giftedness by Pitta-Pantazi, Christou, Kontoyianni and Kattou (2011), as it acknowledges the 'domain-specific' (VanTassel-Baska, 2005, p.358) and multidimensional nature of giftedness (Reis & Renzulli, 2011). Even though this model was developed for the design of assessment tools for upper primary school students, we believe that it can be modified and adopted for preschoolers. This model integrates aspects of: Gagné's differentiated model of giftedness and talent (2003), Renzulli's (1978) model of giftedness and the experiential structuralism theory (Demetriou et al., 2002).

Initially, Pitta-Pantazi's et al. (2011) model adopts the distinction between natural abilities and talent (Gagné, 2003) According to Gagné, the top 10% of age peers in terms of their natural abilities can progressively transform these abilities into talents through a long-term process of learning and training. Pitta-Pantazi et al. (2011) view giftedness as synonym to the term 'talent' and believe that some natural abilities are a prerequisite for mathematical giftedness. In particular, they consider natural abilities to be fluid intelligence, working memory, speed of processing and control of processing.

Pitta-Pantazi et al. (2011) define mathematical giftedness based on the 'Three-Ring Definition' of Renzulli (1978), according to which general giftedness is the interaction between above average ability, creativity and task commitment. Hence, in their model mathematical giftedness is the conjunction of mathematical ability and mathematical creativity.

In the model, mathematical abilities are analyzed into five abilities, based on the five Specialised Capacity Systems (SCSs) suggested by Demetriou et al. (2002). SCSs are sets of specialized abilities with which a person can represent, mentally manipulate, and understand information. They consist of: (a) the qualitative–analytic, (b) the quantitative–relational, (c) the causal–experimental, (d) the spatial–imaginal, and (e) the verbal–propositional system. Mathematical creativity is assessed based on fluency, flexibility and originality (Leikin, 2009). Fluency is the ability to produce a number of solutions. Flexibility refers to the number of different types of solutions reached. For originality the most extraordinary and unique responses are taken into account. Figure 1 illustrates the structure of the model. The factors of mathematical abilities and mathematical creativity constitute mathematical giftedness, while mathematical giftedness depends on natural abilities.

The present study aims at developing a set of tools for assessing mathematically gifted preschoolers aged 4 to 6. These tools are based on a contemporary and multidimensional model of mathematical giftedness.

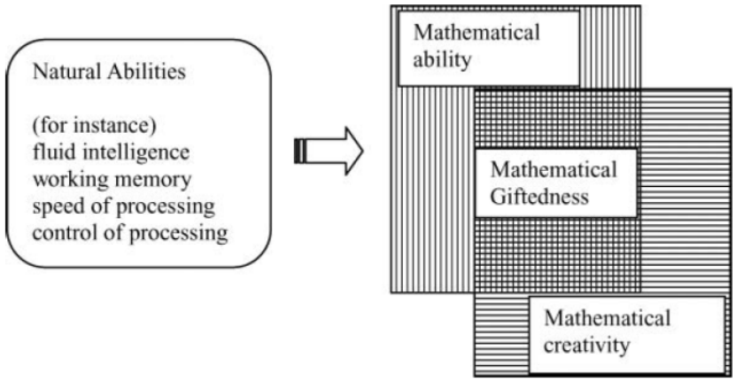


Figure 1: Model of Mathematical Giftedness proposed by Pitta-Pantazi et al. (2011)

METHODOLOGY

Mathematical giftedness assessment tools

The development of the set of tools is guided by the principles for the identification of giftedness described in section 2.1 and the Pitta-Pantazi et al. (2001) model.

(a) Tests: The tests will be administered in 5 sessions of 30 minutes each, through individual interviews, as students might not have developed sufficient reading skills. The tasks will be presented in a digital form so as to maintain children's interest and attention. Short breaks will be provided between tasks to limit fatigue.

(i) Mathematical ability test: It consists of 16 mathematical tasks that can be categorized across the five SCSs (Demetriou et al., 2002).

The qualitative tasks focus on the representation and processing of similarity-difference relations through visual and figural representations. These tasks carry some connotations of the class inclusion tasks developed by Inhelder and Piaget (1958). As shown in Figure 2, in task A, children need to identify similarities and differences among the children of the given picture and recognize the two distinct groups (boys and girls). Then, they should decide which group is the largest.

The spatial tasks revolve around the dimensions of spatial ability proposed by Pittalis and Christou (2010): spatial visualization, spatial orientation and spatial relations. The example task B in Figure 2 assesses the aspect of spatial visualization, since it requires from students to compose and decompose a puzzle, using Tangrams pieces.

The quantitative tasks refer to two key components of number sense: elementary number sense and conventional arithmetic (Pitta-Pantazi, Christou & Pittalis, 2013). Task C (see Figure 2) assesses conventional arithmetic through a story problem. At first, pupils should define the possible amount of cookies eaten, by discriminating the natural numbers greater than 2. Then, they should subtract this amount from 11.

The verbal–propositional system is assessed with 4 tasks, 2 of them examine inductive reasoning and 2 deductive reasoning. For each type of reasoning, one task involves a conventional relationship while the other includes imaginary elements. In task D, children have to use deductive reasoning in an imaginary context. They need to apply the general rule that the Councoun do not fly in the case of Max.

The causal–experimental tasks examine cause-effect relations, hypotheses testing and conclusions drawing. For example, task E requires students to explore a cause-effect relation. They need to specify the sample space of the experiment and then propose a case in which adding 4 balls would increase the probability of picking out a red ball.

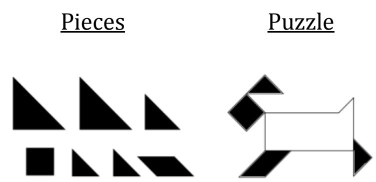
Qualitative-analytic task
The picture below shows a group of children.

Which group of people is the largest?

(a) the boys (b) the girls

(c) the children (d) I can't decide

Spatial-imaginal task
Three pieces are missing from the following puzzle. Circle the missing pieces. (A set of shapes shown below is given to the children, along with the frame of the dog figure).



Quantitative-relational task
Nicole has baked 11 cookies. If she ate more than two cookies, how many cookies are left?

(a) 9 (b) 11 (c) 13 (d) 8

Verbal-propositional task
The Councouns fly. Max is a Councoun, so

(a) Max doesn't fly (b) Councoun don't fly

(c) Max flies (d) I can't decide

Causal- experimental task
Helen has a box that contains 4 balls. She wants to put 4 more balls in it. What colors should the 4 balls be, in order to be more likely to pick out a red ball than a blue one?

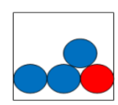
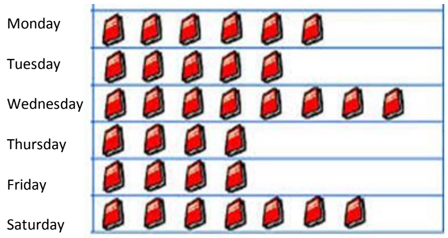


Figure 2: Examples of tasks from each SCSs

(ii) Mathematical creativity test: It consists of 5 multiple-solution tasks, in which students are asked to provide (a) multiple solutions, (b) different solutions and (c) solutions that none of his or her peers could provide. Each task relates to a different strand of Cyprus Mathematics Curriculum. Figure 3 presents two examples of tasks.

(iii) Natural abilities: Fluid intelligence can be measured with the Matrix Reasoning subtest from Wechsler Preschool and Primary Scale of Intelligence–4th Edition (WPPSI-IV), which involves 26 pattern completion tasks. Working memory can be measured using the 35 tasks of Picture Memory subtest, which require children to look at a picture for 3-5 seconds and then identify the same image amongst a group of images. The processing speed can be measured through the 66 tasks of Bug Search subtest, where children are asked to stamp a picture of a matching image within a time limit. Process control can be assessed using the inhibitory control test of Diamond, Kirkham and Amso (2002). A card is presented to the children at a time (totally there are 8 black cards with a yellow moon and 8 white cards with a yellow sun). Children are asked to say the word 'pig' every time they see a moon on the card, and 'dog' every time they see a sun. This test is ideal for children aged 4-4 ½ years old due to the mild level of inhibitory control required.

Task A: The pictograph shows how many books John sold last week.



Tell as many questions as you can, based on this pictograph.

Task B: Classify the figures below into groupsm in as many ways as you can! (Solid figures are given to the children).

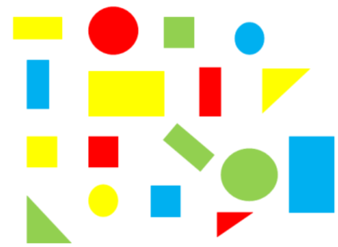


Figure 3: Examples of mathematical creativity tasks

(b) Checklist for educators and children's parents: Children's assessment can be completed using the checklist by preschool teachers and parents, for the two dimensions of Pitta-Pantazi's et al. (2011) model: mathematical ability and mathematical creativity.

The proposed identification process

Keeping in mind that Gagné (2003) refers to the top 10% as talented, our decision leaned towards the use of the 90 percentile rank score as a criterion for identifying gifted students, to allow for inclusive identification. Initially, the children who will attain a score in the top 10% on the natural abilities test will be screened. Secondly, from the children screened, the children who will achieve a score in the top 10% on the rest identification instruments will be identified as mathematically gifted.

CONCLUSION

The goal of the study was to develop a set of instruments for the identification of mathematical giftedness in early years, which relies on a contemporary and multidimensional model for mathematical giftedness (Pitta-Pantazi et al., 2011). From a practical point of view, teachers and curriculum planners may find the proposed set of instruments useful since it enables the early detection of mathematical giftedness, which plays a vital role in children's future development (Pfeiffer, 2002). From a theoretical perspective, the instruments seem to fulfill the principles that have been established for the development of identification tools (Coleman, 2003). Taking into account that the proposed instruments have not been empirically validated, future studies that examine the validity of the proposed set of instruments are highly needed. A question that future research should address is: What is the impact of sociocultural factors on the tools' effectiveness? Furthermore, a longitudinal study should be conducted to examine the predictive accuracy of the instruments.

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ORAL PRESENTATIONS 2.2.

Expertise vs. mathematical giftedness

Chair of the session: **Boris Koichu**

MYTHS ABOUT "GIFTED" MATHEMATICS STUDENTS: HOW WIDESPREAD ARE THEY?

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Abstract. *In the United States, a number of myths about mathematically gifted students inhibit educators, parents and students themselves from developing students' mathematical creativity, expertise and enjoyment. The Common Core State Standards virtually ignore this issue. This paper discusses some of these myths and reports on a few research results from mathematics programs designed to maximize students' mathematical achievement, engagement and innovation. It concludes with a call for information from international researchers in this area to begin an introductory paper for the Topic Study Group on Mathematical Giftedness for ICME 13.*

Key words: mathematically promising, gifted, creative, innovator.

INTRODUCTION

Most states in the U. S. have adopted the *Common Core State Standards in Mathematics (CCSS-M)* in an attempt to raise students' mathematical achievement but the CCSS are virtually silent about the direction to take to increase the numbers, levels, creativity and enthusiasm of our highest performing students. There are a number of reasons that we are not doing a better job of "Preparing the Next Generation of STEM Innovators" as called for in the National Science Board (2010) report by that name, and beliefs in these intertwined myths are part of the problem.

MYTH #1: SOME PEOPLE HAVE A MATHEMATICAL MIND AND OTHERS DO NOT.

The widespread belief in the United States that mathematical geniuses are somehow born that way and that others are born unable to do mathematics at higher levels is perhaps the most dangerous of all these myths. We seem to understand that the Williams sisters, starting when they were four years old, had to spend thousands of hours practicing to become tennis stars and Tiger Woods was practicing his golf swing before the age of two, but we continue to believe that giftedness is a mystical power that people are born with and don't have to work at, and that this is especially true in the field of mathematics. Research by Dweck (2006) and others has shown that students who believe in a "fixed" mindset, that is a belief that you are born with certain "fixed" abilities, do more poorly learning mathematics than those who believe in a "growth" mindset, understanding that your brain changes and grows the more it is challenged to learn. This is true for students who believe that they have a math brain as well as those who believe that they do not.

Experience with scores of students and teachers with whom I have worked over the years, has caused me to ponder whether "school-induced mathematical giftedness" is a possibility. When I was co-principal investigator on a Javits grant, *Project M³: Mentoring Mathematical Minds* (www.projectm3.org), directed by Dr. M. Katherine Gavin at the University of Connecticut, a low-performing Kentucky school was one of the pilot schools in the research study. Before beginning the program, our class of twenty selected "gifted" students' average test scores were at the 22nd percentile in mathematical concepts, the 24th percentile in mathematical problem solving and the 23rd percentile overall in math on the Iowa Test of Basic Skills. After only one year in the program with a heavy emphasis on creative problem solving through reasoning, sense-making, and what we now see as the CCSS-M Standards for Practice, including oral and written communication, the class average went to the 75th percentile in concepts, the 76th percentile in problem solving and the 71st percentile in overall mathematics. Overall, posttest scores did not match those of wealthier, high-performing districts in the study, but they certainly were the highest "value-added" scores of any group. We later followed the *Project M³* program with the NSF-funded *Project M²: Mentoring Young Mathematicians* (www.projectm2.org), which was designed for and implemented with heterogeneous classes of kindergarten through second grade students. Even though these classes, each starting with a wide range of student performance on mathematics tasks, were randomly assigned to either *Project M²* or a comparison group, at the end of a year in the program, more than twice as many students in the *Project M²* classes scored above the mean on an open response posttest as students in the comparison classes, and as many as 7% of the students in the *Project M²* classes scored two standard deviations above the mean as compared with only 0.5% of the students in the comparison classes. (Sheffield, Firmender, Gavin and Casa, 2012)

These results support increasing amounts of evidence from brain research as well as national and international research in teaching and learning mathematics that a wide range of students can learn to perform at far higher levels of mathematics than has previously been thought possible. This corroborates the definition of "mathematically promising" that we developed for the National Council of Teachers of Mathematics that mathematical promise is a function of ability, motivation, belief, and experience, all variables that can and need to be enhanced. (Sheffield et al, 1999)

MYTH #2: GIFTED MATHEMATICS STUDENTS ARE SOCIALLY INEPT AND "NERDY" WHITE MALES

In 1997, I wrote an article for Mathematics Teaching in the Middle School titled From Doogie Howser to Dweebs – or How We Went in Search of Bobby Fischer and Found that We are Dumb and Dumber looking at the portrayal of gifted students in the media and asking for positive media depictions of gifted students and adults, especially females and people of color in STEM fields. After that, we did see Numb3rs, the TV series that ran from 2005 - 2010 that featured the brilliant and attractive Charley Epps and Amita Ramanujan who used mathematics to solve crimes for the FBI, and the Today Show has an online form for parents and teachers to recommend gifted students to be featured on their show, but we also see Sheldon and Amy on The Big Bang Theory who perpetuate the negative stereotype about gifted individuals. When I was a child, it was very unusual to see female lawyers and surgeons in the U. S., but since the influx of television shows and movies portraying these characters in a very positive light, nearly half of the law school and med school students are female compared to only about 5% in U. S. colleges in 1965. Even with billionaires like Bill Gates, Steve Jobs, and Mark Zuckerberg putting their math skills to use in the computer field, the numbers of females in undergraduate computer science and engineering programs remains under 20% in the U. S., with females composing less than 15% of the faculty in engineering. Only about 4% of the engineering students and about 1.5% of the faculty in engineering are black. We should join Miss America 2014, who has a degree in neuroscience, in asking for a reduction in the negative stereotypes and an increase in positive portrayals of both male and female students and adults of all ethnic backgrounds using mathematics in exciting and productive ways. That would go a long way toward encouraging more students to excel in mathematics.

MYTH #3: GIFTED STUDENTS DO MATHEMATICS FASTSER THAN OTHERS AND MAKE FEW MISTAKES

In the 1990s, as part of an NSF Young Scholars Program that I ran for middle school students, Dr. Marlin Languis recorded electroencephalographs (EEGs) on several of the students as they played video games like Tetris and solved a variety of numerical and spatial problems. One of the things we noticed with these brain maps was that the best problem solvers were not the fastest. Their frontal lobes lit up first, indicating that they were planning and thinking ahead before jumping to a solution.

*Mathematicians often spend their entire careers working on a very narrow range of mathematical problems, sometimes working for years on a single problem. The idea that mathematical problems are something to be solved in a few seconds as is often expected in U. S. classrooms seems ludicrous and is counterproductive to the development of mathematicians and others who can become the leaders in the STEM fields of tomorrow. The first CCSS Standard for Mathematical Practice is "Make sense of problems and persevere in solving them." (**www.corestandards.org/Math/Practice/**) We must pose rich problems and give students time to investigate them.*

In 1998, the Clay Mathematics Institute (www.claymath.org) was founded with one of its primary purposes to encourage gifted students to pursue mathematical careers. In 2000, they announced seven Millennium Prize Problems with $1 million allotted for the solution of each problem. These are classic problems that mathematicians have been trying to solve for years. As of February 2015, only one of these problems had been solved and that mathematician turned down the prize. It is likely that some, if not all, of these problems will remain unsolved until students of today are ready to investigate them. Mathematician Dr.

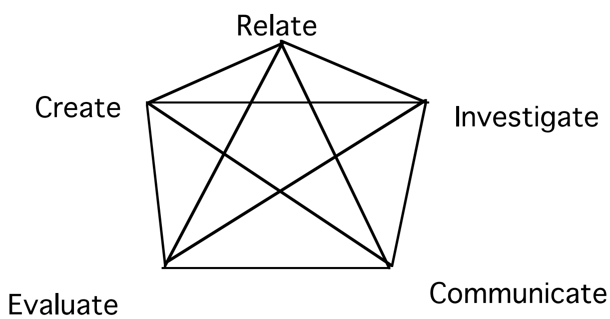
Gordon Hamilton, has taken a number of unsolved problems and worded them in intriguing ways for students of all ages to being investigating. He has a different million dollar problem for each grade from kindergarten through twelfth grade along with a variety of excellent videos, games and puzzles on his website at www.mathpickle.com. These are problems that can be approached by students at any level, and will continue to challenge them in the same way that they engage and intrigue career mathematicians.

It is important that our best students be challenged and encouraged to struggle with difficult problems. Teachers are often afraid that if they let students make errors when solving mathematics problems that students will remember incorrect answers. Research shows that the opposite is true, however. Students who are allowed to make mistakes and "construct viable arguments and critique the reasoning of others" as recommended in the third CCSS Standard for Mathematical Practice have greater enjoyment and a much deeper and long-lasting understanding of mathematical concepts as well as a willingness to attack difficult problems and persevere in their solutions.

MYTH #4: MATHEMATICS IS NOT CREATIVE

Students need to realize that mathematics is a creative, ever-growing subject, and they should not be afraid to take risks with unusual strategies and solutions. Unfortunately, creativity is not mentioned in the Common Core State Standards in Mathematics (CCSS-M) even though it is sorely needed. In the joint publication, *Using the Common Core State Standards for Mathematics with Gifted and Advanced Learners,* from the National Association for Gifted Children (NAGC), the National Council of Teachers of Mathematics (NCTM), and the National Council of Supervisors of Mathematics (NCSM), we suggested that the CCSS-M add a ninth Standard for Mathematical Practice: "*Solve problems in novel ways and pose new mathematical questions of interest to investigate.* The characteristics of the new proposed standard would be that students are encouraged and supported in taking risks, embracing challenge, solving problems in a variety of ways, posing new mathematical questions of interest to investigate, and being passionate about mathematical investigations." (Johnsen and Sheffield, 2012, p. 16)

To encourage innovation, I suggest using a heuristic such as the following that we use in *Project M²*, *Project M³*, and the middle grades mathematics program, *Math Innovations.* It encourages students to persevere in exploring the intriguing depths of a problem, using a variety of approaches and creating interesting new questions to investigate.



(Jensen, 1980)

With this heuristic, students may start at any point and continue to explore at deeper and more complex levels. For example, they might begin by relating the question to something with which they are already familiar and build from there as they investigate the problem. At any point in the problem-solving process, they might evaluate their progress, communicate with others, create new methods of solution or new problems to investigate, and continue to make connections as their investigations take them further into the mathematics. This may seem on the surface to be too time-consuming, but in the long run, it is much more efficient and flexible with long-term, rich understanding, avoiding the need for constant re-teaching, and encouraging the perseverance and complex problem solving.

MYTH #5: GIFTED STUDENTS SHOULD ACCELERATE THEIR MATHEAMTICS CLASSES IN MIDDLE AND HIGH SCHOOL AS MUCH AS POSSIBLE

The number of students in the U. S. taking high school mathematics classes in middle school has exploded since 1990. At that time, only 16% of US students took algebra in eighth grade. By 2013, more eighth graders were taking Algebra I than any other math class. Research by Tom Loveless (2013) of the Brookings Institution found in states where the numbers of students in advanced middle school mathematics courses such as Algebra I increased, the gains on NAEP scores in algebra decreased.

Numbers of students taking calculus in high school in the United States have grown even faster than students taking high school math in middle school. Over the past quarter century, two- and four-year college enrollment in first semester calculus has remained constant while high school enrollment in calculus has grown tenfold. In theory, this should be an engine for directing more students toward careers in science, engineering, and mathematics. In fact, it is having the opposite effect. Too many students are moving too fast through preliminary courses so that they can get calculus onto their high school transcripts. The result is that even if they are able to pass high school calculus, they have established an inadequate foundation on which to build the mathematical knowledge required for a STEM career. Nothing demonstrates this more eloquently than the fact that from the high school class of 1992, one-third of those who took calculus in high school then enrolled in pre-calculus when they got to college, and from the high school class of 2004, one in six of those who passed calculus in high school then took remedial mathematics in college. Some students simply stop taking mathematics classes because they never enjoyed their accelerated classes, and they have met all the math requirements for HS graduation before the senior year of HS or they use AP Calculus scores to meet all the requirements to graduate from college without ever taking a college mathematics course. What the members of the mathematical community—especially those in the Mathematical Association of America (MAA) and the National Council of Teachers of Mathematics (NCTM)—have known for a long time is that the pump that is pushing more students into more advanced mathematics ever earlier is not just ineffective: It is counter-productive. (Bressoud, Camp & Teague, 2012)

How Widespread Are These Beliefs and What Should We Do About It?

In considering these issues, several questions arise, including:

- Are these five beliefs about gifted mathematics students prevalent in other countries?
- If so, are they considered a problem?
- Are there other beliefs that hinder our efforts to engage, challenge, and develop our mathematics students at the highest possible levels?
- What are the most critical practices to counteract these harmful beliefs and to develop students' passion, power, perseverance, creativity and expertise in mathematics?
- What research is there to support these assertions?

It is hoped that this conference might kick-start an ongoing discussion on these and other issues that could form the basis for a seminal paper for the Topic Study Group on Mathematical Giftedness at ICME 13 in Hamburg, Germany in 2016.

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MATHEMATICAL GIFTEDNESS AS DEVELOPING EXPERTISE

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Abstract. *In recent years there have been increased efforts to develop integrative models of giftedness, talent and expertise. Regarding mathematics, Sternberg's approach of developing expertise seems to be particularly promising. In this paper a model of developing mathematical expertise will be presented, some potentials will be described and several reasons for taking this perspective will be provided.*

Key words: giftedness, expertise, developing expertise

INTRODUCTION

It seems evident, on the one hand, that any given (academic) domain has both members that are very gifted in their subject, and those that are less so. On the other hand, it was already in 1916 that William Stern remarked that "[g]iftedness as such provides no more than an opportunity to perform, it is an inevitable precondition, but it does not imply performance itself" (Stern, 1916, p. 110, translated from German original quote). But giftedness does not 'evolve' on its own. Rather, it is based on long term learning processes and practical experience which themselves are reliant on support and stimulating encouragement functioning as inter- and intrapersonal catalysts (cf. e.g. the 'Differentiated Model of Giftedness and Talent' by Gagné (2004)).

Originally, 'giftedness' and 'expertise' were understood as two fundamentally different constructs that have their roots in different research traditions. While 'giftedness' refers to an area-specific potential of high performance, 'expertise' is characterised by continuously outstanding performance levels in a certain domain. However, the above mentioned considerations have recently led to increased efforts to synthesise the two, resulting in integrated models which explain the development of giftedness, talent or expertise.

One particularly interesting approach to such integration is that of a *developing expertise* as put forward by Sternberg (1998, 2000). In the following sections, I will first provide a more detailed description of this model, I will apply and specify it to the domain of mathematics, and then discuss the reasons for its particular suitability to this domain.

DEVELOPING MATHEMATICAL EXPERTISE

According to Sternberg, an individual is in a constant state of developing mathematical expertise whenever he or she is engaged in mathematical activity. In doing so, individuals will demonstrate differences with regard to speed and asymptote of expertise development, which can be attributed to hereditary traits. However, it is also and above all the scope and type of mathematical activity as well as external support inherent in the individual's environment which influences this development. It is assumed that the variability caused by external factors is significantly higher than that caused by hereditary differences (cf. Simonton, 1999).

Decisive for Sternberg's model is that tests aimed at determining one's intelligence or giftedness capture aspects of the current state of expertise development. Such tests are, in practice, often employed to predict one's performance at later stages in life, such as in school, university or professional career. Yet, this by no means implies that the measured constructs are more fundamental, or can be interpreted as underlying cause for the further development from a theoretical perspective.

The basic elements required for developing mathematical expertise are illustrated in Figure 1. Shaded areas indicate the influence of hereditary variables, and the double-headed arrows the manifold interrelations and -dependencies between the four different factors. The dashed arrow symbolises the dynamics of all factors: Their characteristics, the meaning of different factor's attributes, and also the overall significance of the respective factors for developing expertise will change in the course of time.

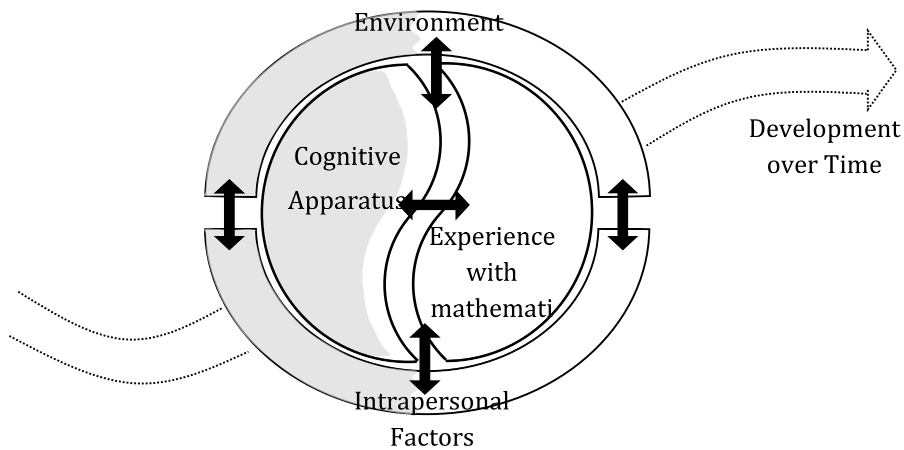

Figure 1: Basic elements required for developing mathematical expertise

Starting point for developing expertise are mainly inherent organic brain structures, and partly domain-specific processes and learning potentials. Regarding elements that are less specific to mathematics, you may consider for instance the amount of grey and white matter, which is genetically determined to a very high extent (Neubauer & Stern, 2007). Other elements include aspects of attention and attentional control, habituation or working memory capacity. Among others, Devlin (2000) made an attempt to denominate basal cognitive potentials related to mathematics, among which he included e. g. the number sense and the ability to think in abstract terms.

Numerous studies on neuroplasticity have shown, however, that organic brain structures remain somewhat flexible even during adulthood and can be influenced by learning processes (e.g. Neubauer & Stern, 2007). Even though the impact of genetic dispositions is comparatively high, the availability of learning opportunities does play an important role.

The key element for the further development of expertise is mathematical activity, as "all intellectual competencies have to be learned" (Weinert, 2000, p. 12, translated from German original quote). Of course it is not only the adequate scope of experience or informal and formal learning practice that play a crucial role in this context, but also the quality of these activities.

While first encounters with mathematics may be of a more playful and informal nature, determined rather by the circumstances at hand, target orientation and effort are important at later stages of education. In this context, research on expertise emphasises the concept of *deliberate practice*. Understood as a systematic and well-organised, frequently guided learning process, it aims to improve one's own performance and features both accordingly demanding challenges that reach respective limits of performance, and constant monitoring of and feedbacks on the demonstrated performance levels. For truly outstanding achievements, it is finally necessary to devote an increasingly large portion of your everyday life to mathematical advancement. This may imply independently looking out for suitable opportunities to do so.

This gives a first indication of the role that the environment plays, i.e. the setting, the interventions and opportunities provided to deal with mathematics, the people and specific (random) events. Bearing in mind Plomin's passive, evocative and active genotype-environment-effects, this realm is also partly subject to genetic influences (Plomin, 1994).
At first, it is family members and then sensitive and dedicated teachers who induce initial steps of development. Later on, master teachers and experts are important as mentors, as they apply specific teaching methods and place priority on performance improvement. Connecting and networking with peers at a comparable level of progress will also become more influential over time, as they provide encouragement, exercise for the mind, guidance, support, inspiration or motivation through competition.

Given that such large quantities of learning experiences are necessary, leads us to conclude that non-cognitive personality traits are of vital importance. A suitable cognitive make-up with a sufficiently large body of experience can only be brought together by combining motivation, performance orientation, determination, perseverance, devotion etc. (e.g. Gagné, 2004). It may at first be up to the parents to exert a certain pressure on the child so he or she will learn the basics and acquire an appropriate attitude to work and according habits (like doing homework and exercises). Yet, the student must, in due time, take on responsibility for his or her own development and progress – a process which requires some self-confidence and soft skills.

This area is also genetically influenced (Ericsson, Krampe, & Tesch-Römer, 1993). Studies examining the interest in mathematics, for instance, indicate that distinct and stable differences already exist among children of pre-school-age, i.e. before they have started formal education (Neubauer & Stern, 2007).

Overall, we may assume that the influence of non-domain-specific characteristics inherent in the cognitive apparatus will decrease in favour of the influence that specific experience with mathematics will have on the latter over the course of time (Heller, 1993). While at first, an individual's direct environment plays an important role in providing supportive conditions for development, the significance of intrapersonal factors grows stronger in the long term.

It is on this basis that the individual will engage in an ongoing process of developing abilities, patterns of action (Kießwetter, 1985), and a specific knowledge base related to mathematics. Given that students can demonstrate extraordinary performance even without extensive mathematical knowledge this area is relatively small in Figure 2. The depiction of mathematical abilities is made with reference to Aßmus (2008), who is developing a system of characteristics for mathematical giftedness among Year-Two pupils – possibly the youngest age group that has been studied to date. By using different types of bullet points in Figure 2, I intend to illustrate that not all abilities need necessarily be equally pronounced. Likewise, there is no simple additive relationship between the respective items (Krutetskii, 1976).

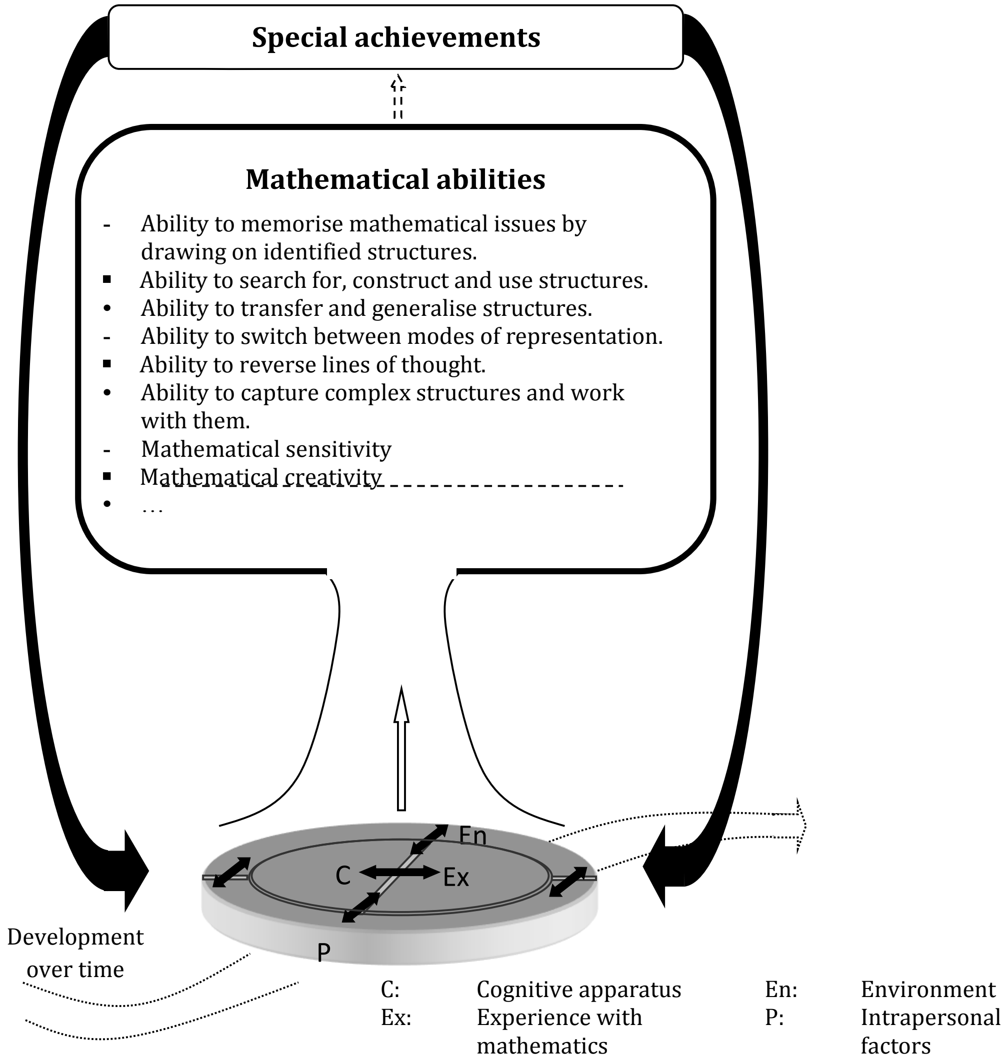

Figure 2: On developing mathematical expertise

It is *possible* that abilities and knowledge base manifest themselves in outstanding achievements, depending on situation and task at hand. This limitation implies not only problems regarding the identification of a developing expertise, but also educational challenges: Through educational measures, the students were also supposed to feel like they were performing particularly well, leading to enhancing feedbacks on the characteristics.

This points once more to the interdependencies between all elements symbolised in Figure 2, as well as their dynamics. The latter applies also to the bold-framed characteristics: As you progress in developing expertise, your subject-specific knowledge base gradually expands and alters, additional abilities may emerge or existing ones can be applied in more sophisticated subject-related contexts.

WHAT'S THIS CHANGE OF PERSPECTIVE ALL ABOUT?

More general arguments for taking on this new perspective on (mathematical) giftedness have been presented in the introduction. Further reasons, related more closely to research findings in the field of didactics of mathematics, include:

- It is undoubtedly that the special abilities and patterns of action used to describe mathematical giftedness are not based exclusively on hereditary traits. Rather, they are largely made possible only by specific experiences and their influence on inherent structures (Wieczerkowski, 1995)
- 'Characteristics of giftedness' are and have generally been shaped on the basis of comparative studies or analyses of work at the level of expertise. To date, there is no empirical evidence that these can in fact be handled and understood as predictors of outstanding mathematical achievements (in a distant future). Neither do we currently know of any adequate theoretical model which explains the way in which the identified characteristics unfold causal effects on someone's mathematics performance.
- In this context there are numerous conformities between characterisations of mathematical giftedness and characteristics of expertise.

Furthermore, the model of developing mathematical expertise has great potential to describe the creation and continuous development of extraordinary mathematics performance of children, youths and adults:

- Individual factors, such as the cognitive apparatus or its genetically predetermined part, will lose their anticipatory nature in this model. This is true also for the current (measured) level of expertise, which does not allow any reliable statements on which level will or can potentially be reached (Sternberg, 1998).
- Rather, the model emphasises that any individual must continuously progress in his or her development through a suitable interplay of all participating factors. This way, he or she has the chance to reach the level of expertise that enables access to fostering programmes etc. and/or lead him to be identified as 'gifted' (Sternberg, 2000).
- Developing extraordinary performance is generally not understood as an autocatalytic process. The environment, in this context, does not merely serve in a defensive function by preventing potential 'disturbances'. Rather, experience in mathematics and thus opportunities to learn are necessary preconditions to develop one's expertise. This also points to the responsibilities of schools and society in supporting necessary long-term and specific learning activities.
- From a pedagogical perspective, inclusive fostering activities should at first be given priority over exclusive ones.
- A (amongst others) cognitive 'basic configuration' is differentiated from mathematical abilities and special achievements. This points to and considers their context-dependence and therewith, for instance, the problem of identifying gifted students or initiating positive feedbacks.
- The model brings to the fore the systemic character of expertise (cf. Ziegler & Phillipson, 2012): the current level of expertise appears simultaneously as an emergent feature and as

an element of a system that is characterised by cross-linkages, dynamics and equifinality, amongst others.

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BAYBURT'S GOT GIFTED

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Abstract. *This paper reports a university-funded project completed in Bayburt, Turkey in 2013. The project was unique for at least two reasons; one being training method followed while the other was its content. During the project, 175 university students and more than 1,000 Grade 5-7 students got training on how to solve Mind Puzzles. Following the training session, a citywide competition was done, and the winning students got their prizes. Developing awareness on Mind Puzzles as well as on their own creative abilities, possible giftedness, seems to be the major outcomes of the project. Research projects to explore the relationships between Mind Puzzles and mathematical creativity have been planned.*

Key words: Mind Puzzles, Competition, developing awareness, seeking gifted students.

INTRODUCTION

Literature suggests that fluency, flexibility, novelty, and elaboration as the main dimensions of mathematical creativity (Calapkulu, 2014; Leikin, 2009; Sriraman, 2005). In the problem solving context, *fluency in thinking* could be understood as the ability to suggest rapid solutions for the problems even if the problems are non-routine problems while *flexibility in thinking* is the ability to suggest as many approaches as possible to a given problem. Guilford (1959) identifies four types of fluency: word fluency, associational fluency, expressional fluency, and ideational fluency. As for flexibility, he points out two groups of flexibility: spontaneous flexibility and adaptive flexibility. Similarly novelty, or originality for some scholars, is described as the ability to propose elegant solutions or at least approaches for the problems. "In general, creativity has been seen as contributing original ideas, different points of view, and new ways of looking at problems." (Torrance, 1988, p. 44)

What seems important for us in defining creativity is the difficulty in its definition, which is quite reasonable. Any attempt to define creativity may limit its meaning and may push the term to the same range of understanding of conformity. However, the beauty of creativity falls into its naturalness and incompleteness. The process definition suggested by Torrance (1988) promises a wide range of understanding and better fits in describing the theoretical framework of our project.

> [C]reative thinking as the process of sensing difficulties, problems, gaps in information, missing elements, something askew; making guesses and formulating hypotheses about these deficiencies; evaluating and testing these guesses and hypotheses about these deficiencies; probably revisiting and retesting them; and finally communicating the results. (p. 47)

Despite the number of use of the first three dimensions, elaboration has been a comparatively less studied topic in the context of mathematical creativity, but not problem solving. For example, Leikin (2009) describes a study on using students' ways to solve problems in order to understand their level creativity.

This paper reports a project, providing opportunities to improve awareness of mathematical creativity of individuals in a specific context. We call this specific context as the Mind Puzzles despite a number of terms used in various cultures. What we mean by Mind Puzzles, or sometimes people use Mind Games as well, is a range of puzzles demanding mathematical thinking to solve. World Puzzle Federation prefer using *puzzles* as a general term to describe aforementioned puzzles whereas some members of the federation also use the word logic (for example, Romania) or mind (for example, Turkey), probably to position themselves apart from crossword puzzles. Similarly, we prefer calling these games as Mind Puzzles, or Mind Games, to avoid confusion. Regarding mathematical thinking, we consider it in this context as developing strategies, ordering, generalizing, specializing, visualizing, representing, developing relationships among a number of representations, and so on (Mason, Burton, and Stacey, 1982; Schoenfeld, 1992).

Given that we collaborated with Turkish Mind Team, which is a member of World Puzzle Federation, we will provide some brief information about these two institutions.

World Puzzle Federation

World Puzzle Federation (WPF) is "an association of legal bodies with an interest in puzzles" whose objectives are basically "to supervise the World Puzzle Championship (WPC), World Sudoku Championship and other WPF events" and "to provide means for an international exchange of puzzle ideas" (WPF website).

Turkish Mind Team

Turkish Mind Team (TMT) is a member of WPF, representing Turkey at the Federation. They have been providing a number of resources for Mind Puzzle fans, competitions at various levels and varying sizes, locally and nationally, and training for those interesting educational institutions and individuals.

Their efforts in education engaged the Ministry of Turkish Education to suggest an elective course at the Turkish middle schools. As a result of their collaborative effort with the Ministry, a curriculum guideline for the suggested course is available now.

THE PROJECT

Undergraduate students attending to the Faculty of Education have to complete some volunteer work as part of their course work. In 2013 spring semester, we invited two other university professors to join us to organize a citywide event. We confronted many challenges while organizing this event even if the city Bayburt is a small city. Funding, training, educational documents, and organization of the educational sessions and final competition were among them.

Funding issue was solved with the support of Bayburt University. Bayburt University provided funding for almost everything, except prizes. We got support for the prizes from local people.

Context

We believe that it is worthwhile to provide some contextual information about Bayburt and Bayburt University to help readers better understand the challenge we had to overcome. Bayburt is a small city located in the northeast of Turkey, with a population of 33,000 people, including university students, slightly more than 5,000. Most of the residents are less educated and live in low socio-economic conditions.

Students coming to Bayburt University are usually under-achievement group of the students based on university entrance exam. Their mathematics and science background is significantly low comparing to the ungraduated students studying in the other parts of Turkey. The answer given by one of the university students to a questionnaire may give an idea to the reader about the level of students: "I have seen Sudoku in some newspapers. However, I never looked at it in detail when I saw the numbers because I have always been very bad with numbers and mathematics."

Only a couple of students out of 175 undergraduates had little information about basic Sudoku, so-called classic Sudoku. None of them knew anything about variants of Sudoku, Kendoku, Tangram, Quarto, or Mangala. This was the first challenge for us because we had to train them in very short period of time and even we were not experts of these games, except the first author, who had been involved in some activities as solver but not trainer. The solution was to get professional help from Turkish Mind Team.

Training as a Model

Given that we had limited amount of money to the training, it was impossible to provide first-hand training for all university participants, a total of 175 students. 50 out 175 were asked to be volunteers for a weekend training program provided by the Leader of Turkish Mind Team, Ferhat Calapkulu. 50 students attended the program on the last weekend of the March, 2013, learned how to solve a number of Mind Puzzles.

Following the training weekend, these 50 students, named as the *education leaders* in the project, were grouped in pairs and assigned to a group of 5 of their peers to train them in solving those puzzles. We assigned each pair of students to 5 students to provide them opportunities to collaborate in case any one of them failed in doing his or her responsibility. They were asked to complete this training session in their spare times in two weeks. We called this process as the *internal training*. Upon completing this internal training period, we had two groups of students with varying levels of expertise and experience: (1) the education leaders, who got training from Ferhat Calapkulu and some teaching experience because they trained their peers, and (2) the *assistant educators*, who got training from their peers and no teaching experience.

After completing internal training, we encouraged students to go to the schools for an *external training*. Since this external training was planned as after school program, some school administrators preferred weekdays while others weekends. The external training program was in two stages: Explaining how to solve Mind Puzzles and Puzzle solving sessions. Education leaders were asked to explain each Puzzle to Grade 5-7 students one at a time at the beginning each session, and then students were invited to solve their own Puzzles in Puzzle solving sessions. These sessions were the sessions that assistant teachers were involved in training. During these sessions, they helped students on the one-on-one basis or in small groups because there were at least two assistant teachers and one education leader in each classroom.

This external training was completed in three weeks. On the last day of external training, school-based competitions were organized to select students for the citywide competition. An important to talk about the training is the way we let them learn –not to teach.

Mind Puzzles Festival

A two-day festival was organized in Bayburt University at the end of May, 2013. During the festival, there were mainly two types of activities: (1) activities open to everyone and (2) the competition. For the festival, we set up ten stations of Puzzles controlled by our assistant teachers. The visitor students, aged 9-14, were introduced to new Puzzles at various difficulty levels. Almost all puzzles were hands-on versions of known puzzles, even classic Sudoku.

Competition

The competition among selected 300 students, 100 from each Grade, was performed simultaneously with the other activities and controlled by Ferhat Calapkulu, the leader of TMT, with the support of 50 education leaders. On the first day, the best 10 out 100 students were won the competition and invited for the final stage on the second day. The second day the first 3 students of each Grade level were selected as the champions of the Bayburt city. The festival was completed by the Prize Ceremony.

Mind Puzzles

In this section, we provide three examples of the mind puzzles we used in the project: Classical Sudoku, Regional Sudoku, and Consecutive Sudoku. All types of puzzles apply the same basic rule. Each row, each column, and each bold boxed area can have each number from 1 to 6 only once, and the goal is to fill missing numbers. For the classical Sudoku, the box is a regular two by three rectangle (see Figure 1).

2		3	5		1
	4			6	
3					2
4					6
	3			2	
6		2	3		4

Figure 1: Classical Sudoku for young learners

We consider this basic Sudoku type is engaging students for associated fluency in thinking (Guildford, 1959). It is because students are expected not only to review the numbers from 1 to 6 but also to consider the rules, not having two same numbers in the same row, in the same column, or in the same box. Although they are supposed to continuously employ probabilistic thinking while working on the puzzle, we do not consider this basic type engages students to think flexibly.

For regional Sudoku (see Figure 2), the box is not a regular rectangle but any shape containing room for 6 numbers. In additional to associated fluency in thinking, this type of Sudoku may engage students for adaptive flexibility in thinking because they are expected to adapt or shift their perception of box from regular rectangle to any irregular shape.

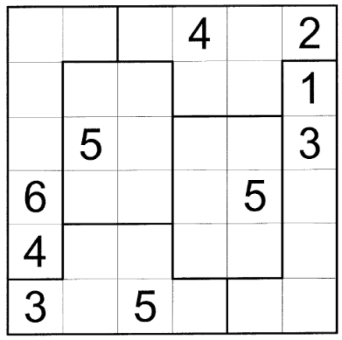

Figure 2: Regional Sudoku for young learners

While solving consecutive Sudoku, students are supposed to place consecutive numbers on each side of the dot (see Figure 3). Therefore, this new type of puzzle encourages students for a new type of adaptation, both associational fluency and adaptive flexibility in thinking.

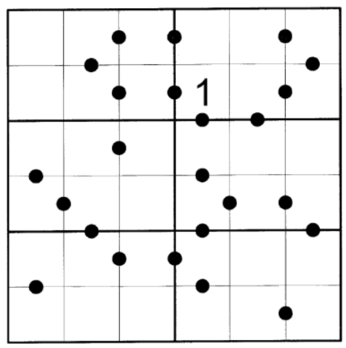

Figure 3: Consecutive Sudoku for young learners

We do not expect the presence of originality in solving Sudoku and its derivatives although there might be some degree of contextual originality in some cases. However, we do believe and expect a great deal of originality in the Escape rooms, which is another Mind Puzzle.

DISCUSSION

We, the authors, would like to re-emphasize that this project was not planned as a research project, but an outreach activity. We started as a fun activity, and the goal was to achieve something might contribute to both parties' –undergraduates and middle school students – cognitive development. In order to understand to what extent we accomplished the goal, we had better look at what has been planning since the project completed.

The first thing we should mention that the activity was repeated the following year and that an elective course, named Mind Games, was created for the undergraduate curriculum for the following years. The course was fully enrolled, and some of the graduates got their first job just because they took such a course in their undergraduate study. Moreover, some of our students have already decided to become experts in this specific area itself and the integration of Mind Puzzles into the regular curriculum of science of mathematics.

We, the authors as the faculty, have decided to apply for national and international grants to explore the relationship between Mind Puzzles and mathematical creativity. In fact, the mathematical creativity as a framework emerged after we decided to take the activity as a research project. The connection was intuitive and unclear enough before we have started reflecting on the content and outcomes of the project. During the activity, we realized that exploring Mind Puzzles from various perspectives and connections to certain topics of mathematics education such as mathematical creativity and problem solving may provide various opportunities for researchers.

Regarding middle school students' gains, we have conducted some informal talks with their teachers, principals, and parents in order to learn what their experience would be. They informally stated that there is a positive shift in students' perspective on mathematics and related academic courses. Besides, one principle specifically mentioned that those who were more engaged in the activities are more eager to take care of daily problems they confront in their school.

What is next?

A brief information about the plans would be as follows: (1) new activities in the other cities, (2) a research project, and (3) national and international working groups to establish for both activity/training and research mathematical creativity point of view.

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ORAL PRESENTATIONS 2.3.

Developing creativity through challenging activities

Chair of the session: **Jong Sool Choi**

CREATIVITY DEVELOPED WITHIN AN ACTIVITY THAT AFFORDS MULTIPLE SOLUTION AND MULTYMODAL ARGUMENTATION

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Abstract. *The present study focuses on students' problems solving processes of one activity from a course, especially designed for third-grade mathematically talented students. The course was designed to foster students' mathematical creativity and reasoning in a problem-solving context. It included 28 meetings over the course of one academic year and interwove problem solving in dyads or small groups, peer argumentation, and teacher-led discussion. We identified the types of solutions, the kinds of reasoning and the kinds of verbal and non-verbal actions (gestures, drawings, folding etc...) used. We show how gestures and other non-verbal actions were interwoven with children's verbal peer argumentation and led them to new insights on the concept of area.*

Key words: multiple solutions, multimodal argumentation, geometry, area, young mathematically talented students.

INTRODUCTION AND THEORETICAL FRAMEWORK

The work with mathematically talented students should afford opportunities for challenging high-level cognition. Such students need to be engaged in tasks that are complex, that invite students to raise questions, to make conjectures, to argue in order to explain, clarify, and revise their mathematical ideas and problem solving processes.

Mathematically talented students need to develop their creativity, and since creativity is at the same time defined and identified through fluency, originality, and flexibility (Silver 1997), the tasks students cope with should also encourage them to generate many solutions, and to provide rich and various kinds of reasoning strategies to their solutions. While coping with the above tasks it is recommended to encourage the student to experience the generation of productive argumentation processes. Researchers showed that argumentative talk may lead to conceptual learning and change (Asterhan & Schwarz, 2009).

In research works concerning argumentative talk we may often find some use of Toulmin's model for analyzing and documenting interactional argumentation processes (Toumin, 1958). We used a simple version based on Toulmin's scheme for analyzing individual and collective mathematical arguments, while investigating a dyad's common work. Our scheme for an argument structure in this research is presented in Figure 1. Reasoning is a common term to the explanation-justification-proving of the relationships between the Data/Ground and the Claim/Conclusion.

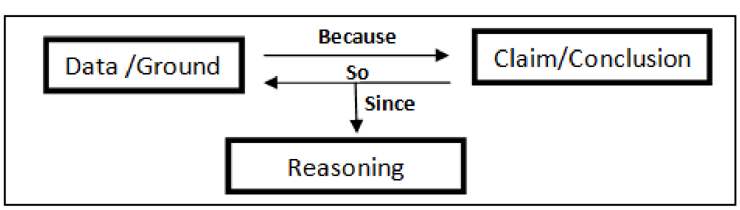

Figure1: A scheme for an argument's structure used in this research

Recently mathematical activity is recognized as based also on multiple channels of communication – gestures, drawings, bodily actions, or manipulation of software. These channels of communication provide precious information on mathematics learning. As Radford (2009) claims: "Mathematical cognition is not only mediated by written symbols, but that it is also mediated, in a genuine sense, by actions, gestures and other types of signs" (p. 112). In this paper we demonstrate the complementary role of the design of tasks for facilitating productive argumentation and developing mathematical creativity**.**

THE STUDY

Research goals

(1) To design sequences of tasks that stimulate deliberately planted socio-cognitive conflict leading to the production of multiple solutions, multiple types of problem-solving strategies, and justifications in multi-channeled argumentation.

(2) To examine whether the design leads to emergence of understandings the area concept.

The course

Three groups of 20 talented 3rd graders participated in a special enrichment program in mathematics over three successive years. The program was designed to develop problem solving activity, in which the tasks encourage multiple solutions, strategies and practices such as drawing a diagram, adopting trial and error methods, identifying patterns and putting into action deductive considerations. About 25% of the activities dealt with issues related to the geometrical concepts of area and perimeter and their relationships. The activities for the course were designed to trigger productive argumentation and creativity. Five designed principles were adopted: (a) inviting to produce multiple solutions, (b) creating collaborative situations and (c) socio-cognitive conflicts, (d) providing tools for checking hypotheses and (e) inviting to reflect on solutions.

The 'sharing a cake' activity as a principal research tool

The activity was designed to facilitate an understanding of the area concept and, in particular, the fact that shapes may have equal area without being congruent. Figure 2 presents a shortened version of the activity. The goal of the first task is to encourage students to provide diverse solutions and diverse justifications. Grid squares representing the cake were given to students in their worksheets to encourage them to find many diverse solutions, and to provide a proper context for comparing areas of various shapes (especially non-congruent) created on the square grid (a tool for checking hypotheses). The goal of the second task is to trigger a cognitive conflict in which a non-congruent partition of the square is presented. Moreover we introduced a text by virtual student Mindy who raised the common misconceptions regarding the concept of area.

Methodology

The course students were selected on the basis of a recommendation letter from their teachers and a test. The lessons of the course were videotaped with four cameras. The first camera was directed at the teacher at the beginning and the end of each lesson. Two of the other cameras documented one dyad or a small group of three students. The fourth camera was moveable and documented the interactions of dyads and triads. We analyzed transcriptions of the videotapes, to identify instances of students using semiotic resources such as: gestures, drawings, bodily actions manipulating with artefacts, and use of language (Radford , 2009). In addition, we analyzed the worksheets from the three successive years using quantitative statistical methods.

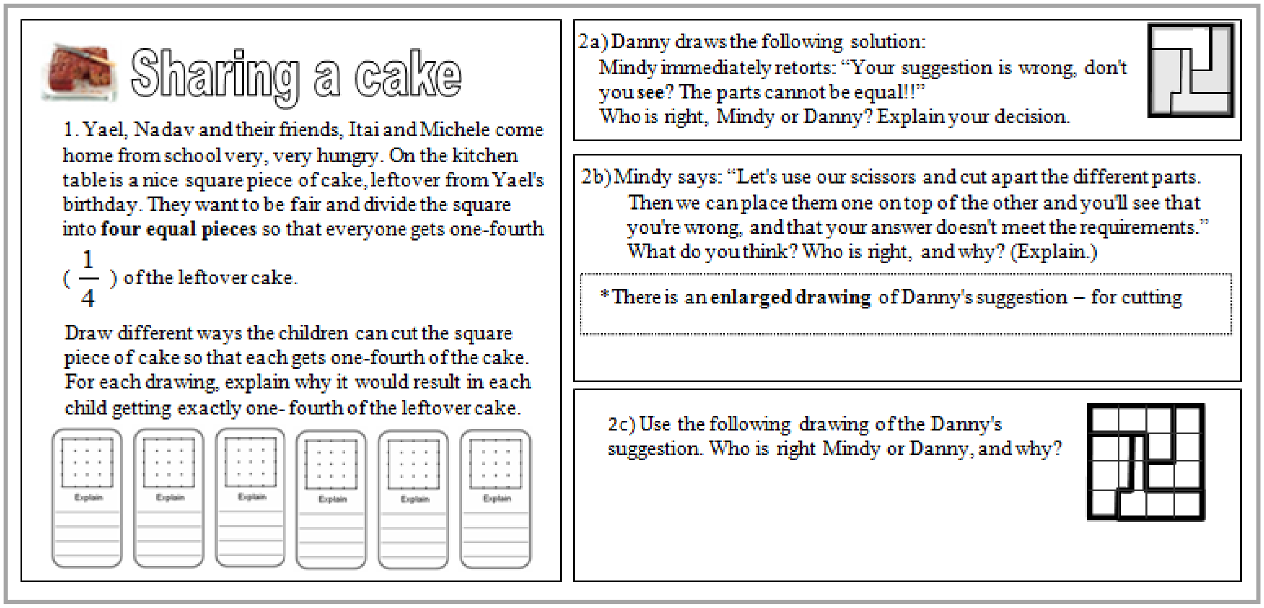

Figure 2: A shortened version of "Sharing a cake" (Task 1 Task2a, 2b and 2c)

Data and data analysis of "Sharing a Cake" activity

In this article we mainly discuss the findings of the second task: 2a, 2b and 2c. But first we present briefly the findings of the first task (Prusak, Hershkowitz, & Schwarz, 2013). We analyzed 38 worksheets done by dyads or small groups, from three successive years. We found multiple and creative solutions. We identified four main types of solutions (see Fig. 3). In Type A, all four shapes were congruent and were created by simple partitions: drawing diagonals, perpendicular bisectors, or segments parallel to one side. 95% of the students proposed all three Type A solutions. In Type B, all four shapes were congruent but they were created with more sophisticated partitions. Type C solutions consisted of two different pairs of congruent shapes. And in Type D, there were no more than one pair of congruent shapes (quite often all four shapes were non-congruent – see Fig. 3). We found that 84% of the students produced at least three different types of solutions. This means that they produced at least one solution in which not all of the four parts were congruent. In addition to the solutions drawn on the worksheets, the analysis of the videotapes revealed that these solutions were the result of rich interactions during which students justified their solutions and convinced their peers in various ways.

In the analysis of the students' worksheets we identified three types of justifications: (1) congruency-based (2) compose and decompose, which might lead students to the idea of equal area non-congruent figures and (3) counting area units. We analyzed the written justifications and found that 46% of the justifications were congruency-based, 42% were counting justifications, and only 12% were compose-and-decompose justifications. In light of the findings gathered from Task 1, we turn to analyze the findings of Task 2.

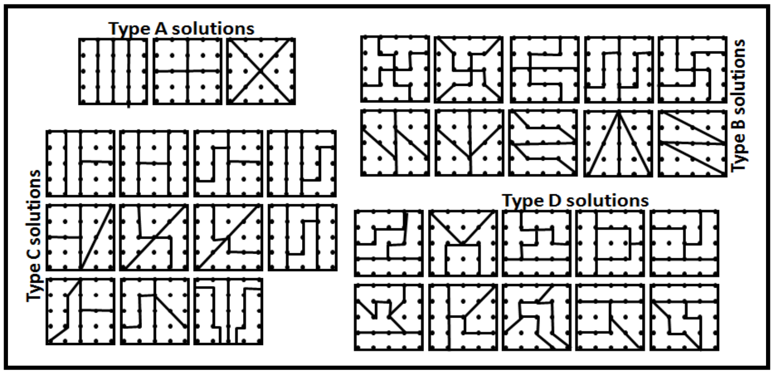

Figure 3: The four solutions' types of Task 1

The second Task (2a, 2b, 2c)

In this paper we present a case study of one of the dyads that participated in the course. We present the way we analyze the argumentation processes generated. Due to the length restrains we present only parts of the rich dialogue.

Task 2a

Liron 4: I think that Danny is right the four parts are equal

Hod 5: Only two parts are equal. [Points at 1 and 2 with two fingers, stares at drawing, then shows with two fingers that he measures the length of a first segment in part 2 and then measures the same length on the corresponding segment in part 1 (See Figure 4a). [Hod draws squares to divide each part to square units] (See Figure 4b)

Inter. 6: Yes, so you claim that Danny is wrong?

Hod 7: There are more squares so this is larger [pointing on the squares partition he drew on part 1, 2&3. pointing with his finger moving around the boundary of part 3.]

Hod 8: They are equal [puts one finger on part 1 and a second finger on part 2] but this one [points to part 3] Not! [Not equal to part 1 or part 2] (See Figure 4c)

Inter. 9: What about the last part? [4]

Hod 10: But part 3 is already not equal [points to part 3] so they cannot be the same as the others!

Inter. 11: What do you think, Liron, do you think that all the parts are equal?

Liron 12: Yes because here it is half [composes parts 1 +2] and here is also half... (See Figure 4d)

Hod performs a new partition and claim (See Figure 4e)

Hod 13: No, I said no! [Meaning that he still insists that the parts are not all equal]

We present some of Hod's arguments using the scheme (see Figure 5). We can see that at the beginning Hod insisted that Danny was wrong and reasoned his claim visually with a measuring gesture (Hod 5) and by counting the square units he drew (Hod 7). Hod 8 claimed that part1=part2 but part 3 is not equal to neither of them. When the interviewer asked him about part 4 he firmly declared: (Hod 10). But part 3 is already not equal so they cannot be the same as the others, which is a beginning of deductive reasoning.

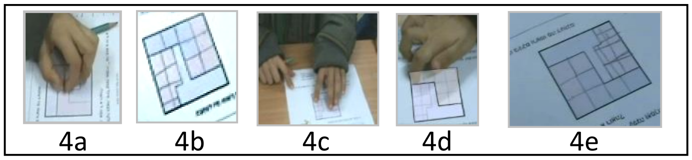

Figure 4: Multimodal actions in task 2a

Liron, on the other hand, claimed that Danny was correct. He referred to (parts1+part2) as half of the square and (part3+part4) as the other half, but he thinks that this is enough data to claim that each part was one-fourth of the square (Liron 12). We can see that both students used logic reasoning (Hod 10, and Liron 12). Liron insisted that Danny is right and then demonstrated the accuracy of his claim by cutting the enlarged drawing of Danny's suggesting, and thus "proved" that parts 1 and 2 were equal to parts 3 and 4 (Liron 18).

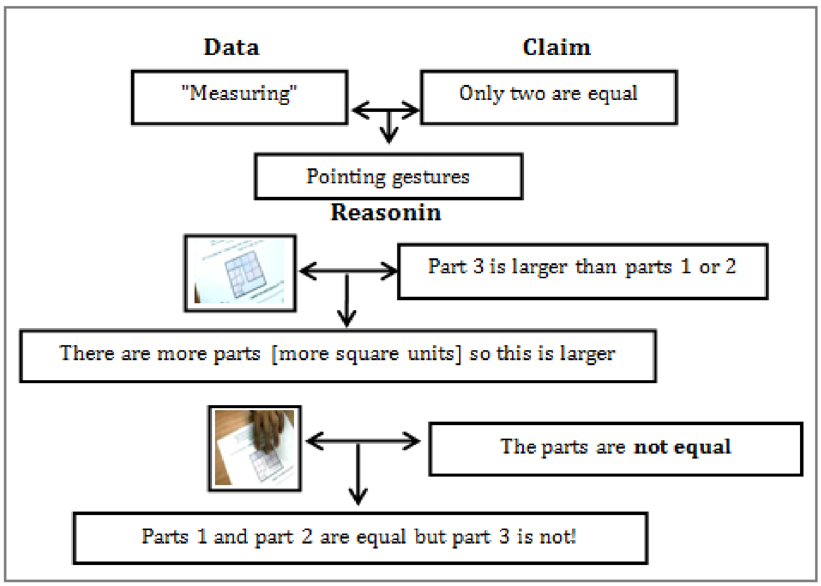

Figure 5: Some of Hod's arguments in the argument's structure scheme

Task 2b

Liron 18: because it seems to me like a half that way [pointing with Encompassing Gesture]. [Liron brings enlarged drawing of Danny's suggesting and cut it]

Liron [turns and puts the (part 1+part2) on (part3+part4) to show that they are congruent]

Hod 23: You convinced me.... Oh no, not really convinced me. It's bigger than that! (Part 3 is bigger than part 4).

Hod 24: [Looks carefully at Liron's efforts (Liron 22)] I have an idea. [Hod cuts part 4 and reassemble it to look exactly like part 3.] Danny is right!

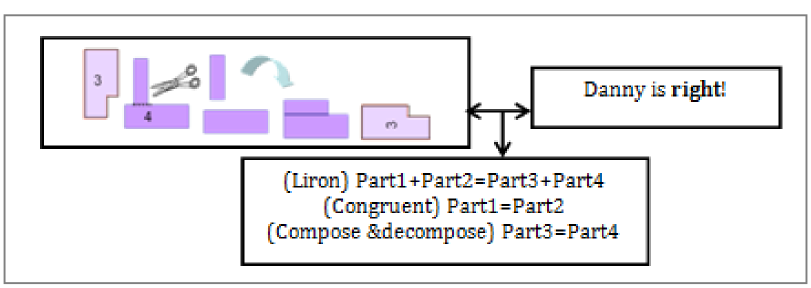

Figure 6: Hod's argument's structure scheme scheme

Hod (24) found a way to decompose part 4 and recompose it as if parts 3 and 4 were congruent. The "new" parts 3 and 4 are equal. The thinking processes Hod underwent in Task 2b interweave creative ways: visualization, counting, compose and decompose and deductive reasoning: (See the argument's structure scheme in Figure 6).

Hod (5) part1=part2 and part3>part 1 & part3> part 2 (*Visual* reasoning)

Hod (23) part1+part2=part3+part4 *visualized* after Liron 22 (reasoning by cutting)

Hod (24) part3=part4 *compose and decompose* by cutting

Hod (24) ⇒part1=part2=part3=part4 *deductive reasoning*

(Each part is a half of half square so is a one fourth of the square)

Task 2c: Not all dyads were able to conclude this area equality in the 2b stage; therefore the goal of Task 2c is to provide the students an easier tool for testing hypotheses with square grid. And indeed about 20% of the dyads agreed upon the right solution only after coping with Task 2c.

We illustrate the learning process generated in the interaction between peers that was initiated by a socio-cognitive conflict: Liron claimed that Danny is right, and he almost convinced Hod. But Hod felt that the chain of reasoning Liron gave was incomplete. But, at last, Hod, who started by claiming that Danny was wrong, eventually completed the "proof" concerning correctness of Danny's solution. We cannot imagine this shift occurring without such multimodal interaction with a peer.

CONCLUDING REMARKS

The design of activities such as "Sharing a Cake" afforded collaboration and experiencing problem solving processes which led to many solutions and to various types of justifications. And indeed the analysis of data in Task 1 shows that the students produced many surprising solutions and justifications (See figure 3). Most of the students understood at the end of Task 1 that incongruent figures might have the same area. The socio-cognitive conflict purposefully designed in Task 2a, b, c triggered the enactment of multimodal argumentation processes with the use of non-verbal actions that sometime palliated the difficulty to articulate verbal justifications. With the help of these multiple channels, we observed the seeds of deductive considerations. We believe that the elaboration of the methodology we used could be refined and validated in future research.

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“MATHE FÜR KLEINE ASSE” – AN ENRICHMENT PROJECT AT THE UNIVERSITY OF MÜNSTER

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Abstract. *At the University of Münster, projects that foster mathematical talents have a twenty-year-old tradition. Currently, the long-term enrichment project “Mathe für kleine Asse” is the established design based on many years of experience and having already produced a lot of both research results and practical recommendations for teachers. Beyond fostering mathematical talents and beyond scientific activities, the project is imbedded into the education of teacher students, since they can take part and learn exemplarily how to diagnose and foster mathematical talents by practical work with children guided and accompanied by theoretical courses. In this article, important aspects of the project’s design will be reported and discussed.*

Key words: mathematical talent; mathematical giftedness; diagnostics; support programs

INTRODUCTION: GENERAL INFORMATION ABOUT “MATHE FÜR KLEINE ASSE”

“Mathe für kleine Asse” (a metaphor like “Math for small pundits”) is an enrichment project that was founded by Friedhelm Käpnick in 2004. The design referred to in this article mostly focuses on children between the 3rd and the 6th grade. Additionally, there are some blended learning courses for children of the 7th or 8th grade as well as pre-school courses. Every year about 150 children of partner schools in and around the city of Münster visit the project sessions that take place at the university of Münster in a room that is prepared as a learning workshop where, e.g., a lot of working materials can be found. The aim of this article is to outline the projects constitutive theoretical framework, its purposes, its applied diagnostics procedures, types of organization of project sessions, and, finally, notes about qualitative aspects of evaluation (for descriptions of the project’s design see also Käpnick, 2008; Fuchs & Käpnick, 2009; Benölken, 2011).

THEORETICAL FRAMEWORK OF MATHEMATICAL TALENT – AN OVERVIEW

Concurrent approaches that try to describe the phenomenon of mathematical talent indicate a scientific consensus on the following aspects (Käpnick, 2013): First, the phenomenon is complex, and it demands the consideration of both cognitive and co-cognitive intra- and interpersonal determinants (see also Käpnick, 1998). Second, it occurs domain-specifically – for instance, particular criteria of “mathematical talent” have been emphasized (e.g., Kießwetter, 1985; Käpnick, 1998). Finally, “talented” children should be identified and fostered as early as possible to support the development of their potentials. As a consequence, “talent” has to be seen as a dynamic phenomenon that demands a holistic view on individual personalities and, therefore, complex long-term process-diagnostics. The mentioned aspects are synthesized within the approach of Fuchs & Käpnick (2009), who constructed a model that describes the development of mathematical talents at primary school age and that provides the base of diagnostics procedures of “Mathe für kleine Asse” (fig. 1; an analog framework is used with the groups of the 5th and 6th grade; see Fritzlar, Rodeck & Käpnick, 2006). In its center, it includes a system of criteria that operationalizes mathematical talent with children of the third and fourth grade by Käpnick (1998).

Therefore, "mathematical talent" is seen as an above-average potential as to the criteria of Käpnick. This potential is characterized by individual determinants and a dynamic development depending on inter- and intrapersonal influences in interdependence with personality traits supporting the emergence of talent.

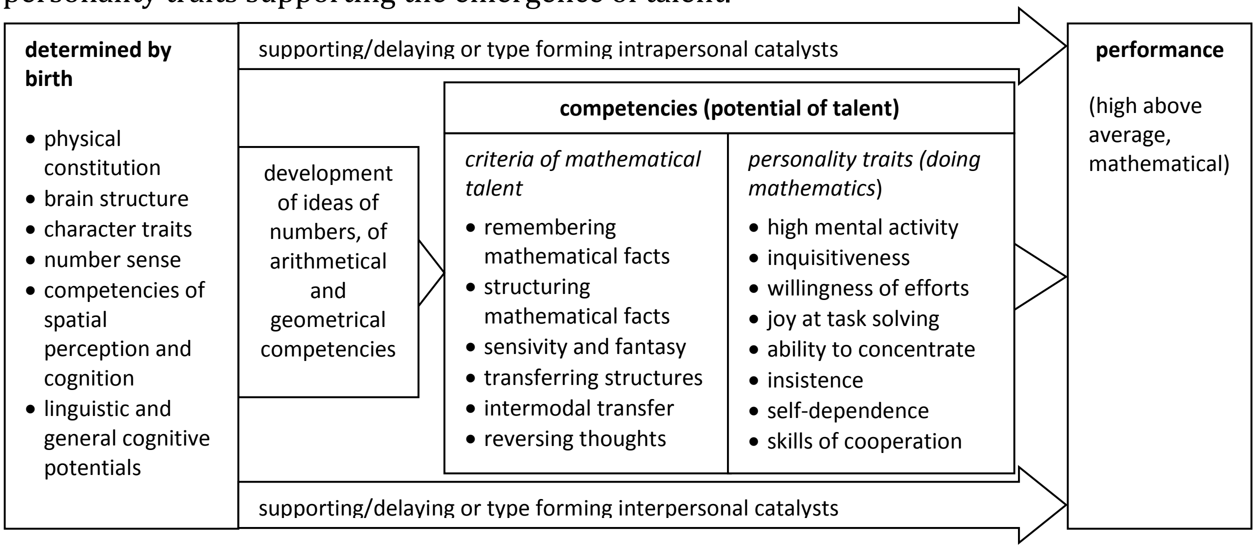


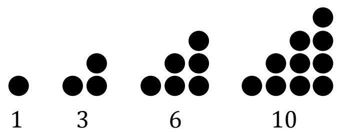
Figure 1: The model of Fuchs & Käpnick (2009), reduced and translated by the author.

PURPOSES OF THE PROJECT

The project is imbedded into the education of teachers at the university of Münster. Therefore, teacher students take part in processes of fostering the children and diagnosing their individual mathematical potentials. As a conclusion, the project's purposes cover three dimensions: (1) As to the children, their joy at task solving, playing with numbers or geometrical shapes and, e.g., their intellectual inquisitiveness should be fostered by enrichment of standard mathematical lessons. On that base, the developments of their entire personalities should be supported. Finally, children should gather an adequate imagination of mathematics and activities of mathematicians. (2) Regarding the education of teachers, students should in particular get experiences and abilities of diagnosing and fostering mathematical talents. Furthermore, working in the project means to be close to scientific research, so they are able to collect experiences in this field, too. (3) As to research, main purposes are the perpetual evaluation of the theoretical framework by, e.g., complex case studies (e.g., Fuchs, 2006; Benölken, 2011), the development of criteria systems of mathematical talent (e.g., Ehrlich, 2013) or the development of diagnostics tools (e.g., Berlinger, 2015). With regard to the practical use of the project, methodological recommendations as well as fields of mathematical tasks should be developed, especially in order to organize similar courses of fostering mathematical talent at schools (e.g., Benölken, 2013; Fuchs & Käpnick, 2009; Käpnick, 2001).

PROCESSES OF DIAGNOSTICS AND APPLIED TOOLS

Corresponding to the above outlined view on "mathematical talent", diagnostics are organized as a long-term process: As a first step, at the beginning of the third grade, teachers of schools in Münster elect children corresponding to Käpnick's criteria and suggest a participation in the project. In a second step, children can visit the project to get to know its organization and atmosphere. In a third step, they have to fill in a half-standardized introductory test (organized as a competition) containing "indicator tasks" that operationalize Käpnick's criteria. Simultaneously, the process-diagnostics begin and continue as long as the children take part in the project considering both cognitive and co-cognitive parameters. Therefore, both (half-) standardized tools like tests that are similar to the introductory one (see the example of fig. 2) or amending IQ-tests (e.g., the "CFT-20") as well as non-standardized tools like observations on children's task solving using rating sheets (e.g., Fuchs, 2006), interpretations of video documentations' transcripts or guided interviews (e.g., Benölken, 2011) are applied. In this manner, an impression of the children's individual talents gradually emerges.



The triangles of circles will be increased regularly. Give reasons for your answers to the subsequent questions. (a) How many circles does the next triangle contain? (b) How many circles does the triangle whose lowest row consists of 30 circles contain? (c) How large is the sum of all numbers between 1 and 30?

Figure 2: An "indicator task" focusing on structuring and intermodal transfer (Käpnick, 2001).

ASPECTS OF ORGANIZATION OF PROJECT SESSIONS

The children are grouped by their affiliation to the 3rd, 4th, 5th or 6th grade. Thus, usually between 20 and 25 children take part in each group. Each group meets every two weeks in the afternoon during the university's terms, i.e. about 20 times per year.

Because of the project's implementation into the education of teachers, all teacher students who participate in the project have to learn basic information about theoretical findings on (in particular mathematical) giftedness and talent as well as conceptions of diagnostics and fostering by taking a specific theoretical course before. Every project meeting is divided into three parts: First, a preparing workshop where teacher students and guiding scientists come together for 30 minutes, e.g., in order to preview the sessions' sequences or to emphasize specific aspects of observation. Then, the children's session starts, which takes 90 minutes. Finally, a reflecting 90-minute workshop takes place where teacher students and scientists review, e.g., the composition of the applied task fields, possible success or failure of sequences and, of course, notes of every child's mathematical talent.

Themes and task fields that are applied in the project's sessions of each group follow a long-term scheduling that is composed for an entire school-year. Beyond mathematical excursions to different places in the city of Münster once per school-year (see for details Berlinger, 2012), beyond testing sessions that are organized as mathematical competitions two times per school-year and beyond some special-issue-sessions (like offering different thematically independent mathematical puzzles or presentations of professional mathematicians), the work on complex task fields is the main form of methodological organization: No more than one theme per session should be introduced considering the idea of natural differentiation as much as possible. A task field in that way should consider some of these aspects (see also, Käpnick, 2001): (1) Children's inquisitiveness and joy of problem solving should be attracted. (2) The mathematical substance should be adequate, i.e. each child should – corresponding to his or her potentials – be able to discover interesting facts that might reach a real deepness. (3) The task field should be open regarding the applying of different working materials, different styles of solutions or different ways of solutions' presentation. (4) Finally, it should be possible to find further problems emerging from the introductory ones which can be explored by the children.

Similar to common school lessons, the schedule is divided into three stages: In the beginning, a problem is offered that leads to the core of the session's theme within 5 to 15 minutes. Subsequently, children turn to researching activities for about 60 minutes, while they can organize themselves as to social modes of working, applied working materials or ways of describing their solutions while the guiding scientists or the teacher students act in the background, just giving impulses on self-help to the children. Finally, the children present and compare their solutions; the correctness of results, however, is not essentially more important than consistent ways of argumentations or the like.

A representative example of task fields is the theme "intersection of lines" (German translation from "Schnittpunkte von Geraden"; e.g., Käpnick, 2001; Fuchs & Käpnick, 2009) that usually is applied in the fourth grade group of the project. In the first stage, terms like "line", "interception" or "parallel" are discussed to ensure similar backgrounds of knowledge with the children. Afterwards, the children are questioned how many intersections can appear by drawing four lines. Thus, the discussion focuses on a first example that leads to the emphasis of four prototypes (fig. 3) which can be useful in subsequent research activities. Finally, the main research question is proposed – which is often done by children who simultaneously provide corresponding hypotheses: *How many intersections can appear by drawing one, two, three, four, five, six or more lines?*

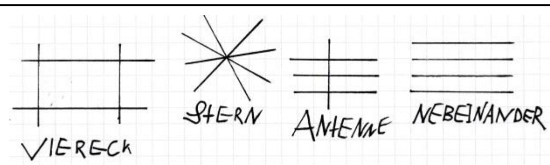


Figure 3: Prototypes of intersections of four lines.

In the second stage, children work on the research questions while decisions about social working modes and forms of presentations of solutions are up to themselves. In this context working materials like pick-up sticks, ropes or white paper to fold solutions are offered for their consideration (but some children are content only to create formal symbolic solutions with tables or the like). Moreover, the children are told to produce comprehensible solutions in order to present their results using, for instance, wallpapers.

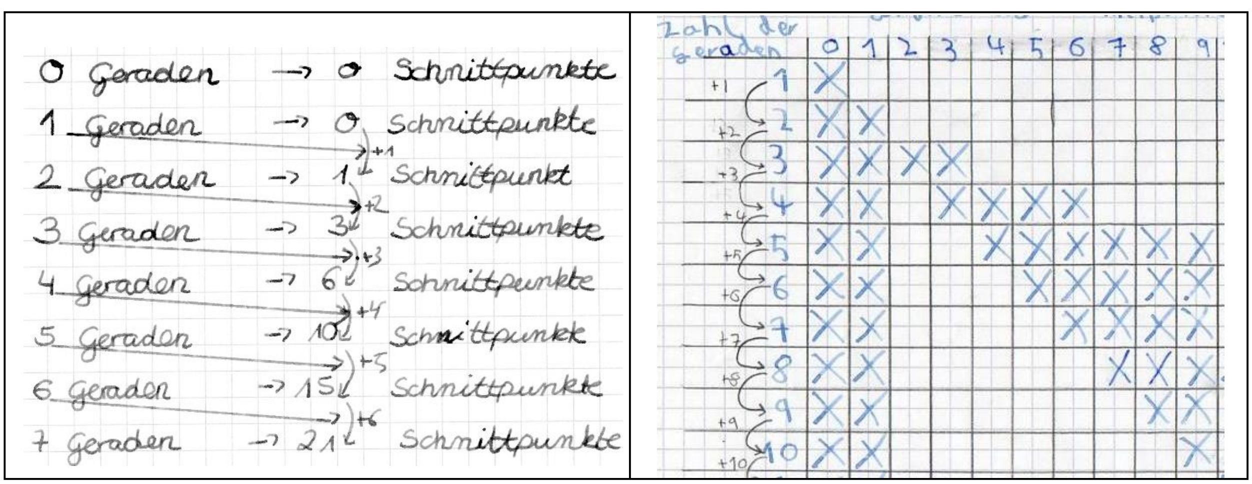


Figure 4: Children's solutions on the research question of intersections of lines.

In the concluding third stage, results are discussed and compared. From experience, there is a wide range of solutions' deepness: On the one hand, relatively simple observations are presented that discover the system of maximum-intersection-number-increasing from "handmade" solutions of a certain number of intersections (fig. 4, left). On the other hand, there are more complex solutions that use, e.g., tables to decline hypothesis on all possible intersections numbers with higher numbers of lines (fig. 4, right). Occasionally (especially, however, with older students), a hypothesis on a formula like $x=n\cdot(n-1):2$ is proposed that allows to calculate maximum numbers of intersections x directly by the number of lines n. Thus, it is obvious that there are a couple of further questions that might emerge from the discussions.

NOTES OF (QUALITATIVE) EVALUATION

Although "Mathe für kleine Asse" has been realized for about ten years, a systematic long-term evaluation has not been processed yet, in particular as to quantitative measurements. In contrast, there are some qualitative results that can be adduced in this context.

First, regarding *the theoretical base* of the development of mathematical talent, during the past years many case studies on different focuses of research have been made which exemplarily, e.g., confirm important aspects of its appropriateness like the specific characteristic of mathematical talent or the need of a holistic view on questions of both diagnostics and support, especially with young children (e.g., Käpnick, 1998; Fuchs, 2006; Benölken, 2011). Moreover, as results of many years of experience and perpetual further development, the above-mentioned instruments of diagnostics can be seen as approved in particular in regard to their use within long-term process diagnostics.

Then, as to the *design and organization of the support program* that is based on the theoretical approach, teacher students consistently report positive effects on their education, especially as to the development of skills in diagnostics and fostering, which were experienced from the field of mathematical talent, but which were useful beyond this context, too. Furthermore, many students describe their practical work with children as one of their education's highlights, since similar opportunities are rarely included in the first years of their academic studies (for examples of quotes see, e.g., Käpnick, 2008). Thus, implementing a project like "Mathe für kleine Asse" into the education of teachers has to be assessed positively (comparable assessments can be found about similarly organized projects focusing on children with low arithmetical competencies, see, e.g., Benölken & Kelm, 2015).

Finally, as to *the project's usefulness in the support of mathematically talented children*, on the one hand, there are notes of positive effects on behavior in mathematical school lessons, when children take part in the project. For instance, girls often seem to develop more advantageous characteristics of motivational factors like self-concepts (e.g., Benölken, 2014) – similar phenomena are reported by teachers again and again. On the other hand, at the end of each school year, we apply a short qualitative questionnaire asking children to take stances about their evaluation of the project sessions and their participation in the following school year: in large parts, children usually assess different factors of the project's design, organization or session contents quite positively (for examples of quotes see Fuchs & Käpnick, 2009; Käpnick, 2008).

Altogether, the synthesis of the mentioned evaluation perspectives indicates that there are various important aspects of the organization and design of "Mathe für kleine Asse" which have to be assessed positively. More systematic evaluations are scheduled and the results will probably be presented during the conference in honour of the project's tenth anniversary in April 2015.

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XCOLONY EDUCATIONAL ACTIVITIES ENHANCE SPATIAL REASONING IN MIDDLE SCHOOL STUDENTS

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Abstract. *This study aims to measure the effectiveness of the manipulative learning tool XColony KDK for developing spatial intelligence as a component of mathematical creativity in students. Our study engaged two groups of 5th grade students: the target group followed through a KDK program of 8 weekly sessions of 1 hour; the control group followed a similar educational program, except that the 1 hour sessions were focused on standard topics instead of KDK. Students in both groups were tested with specially designed tests at the beginning and at the end of the KDK program. Target group data was calibrated based on the control group measurements to eliminate biases induced by curricular learning, variability in difficulty level across test problems and other external factors. We found that in the target group XColony KDK activities lead to a significant 26% increase in spatial reasoning and to 17% increase in a global score for comprehension, problem solving and reasoning.*

Key words: creativity, manipulatives, spatial intelligence, XColony KDK

INTRODUCTION

The mathematics community widely accepts the idea that the essence of mathematics is not limited to problem solving but it also includes the creative thinking process that led to finding the solution (Dreyfus & Eisenberg, 1996; Mann, 2006). Consequently, promotion and development of creative mathematical thinking in school becomes a necessity (Leikin & Pitta-Pantazi, 2013).

In many countries, this necessity is included in the education policies. The Romanian National Curricula for secondary school recommends the development of "open and creative thinking" (MECI, 2009). In the USA, similar actions are recommended by the NCTM Standards (NCTM, 2000). On the other hand, teachers perceive the existence of some "barriers" in organizing creative mathematical activities in school (Kattou, Kontoyianni & Christou, 2009).

One can ask: *What activities could teachers use for developing creativity in students?*

Traditionally, creativity is assessed by fluency, flexibility, and novelty parameters (Torrance, 1974). Recent studies take into account the *solution space* parameter to assess mathematical creativity (Leikin, 2007). Problem solving and problem posing were found to foster creativity (Pehkonen, 1997; Silver, 1997), and the use of "physical" manipulatives (Brunkalla, 2009), or "virtual" manipulatives (Bos & Lee, 2014) develop students' creativity in geometry. On the other hand, a recent study (Kell, Lubinski, Benbow, & Steiger, 2013) shows that spatial intuition predicts innovations in STEM (science, technology, engineering, mathematics) domains.

We focus our study on the following questions: *How can we incorporate manipulatives into mathematical classroom activities, in order to foster creativity and visual perception, as well as mathematical thinking in a geometric context?*

METHOD

Our empirical study is based on the classroom use of the educational resource called the *Knowledge and Discovery Kit* (KDK) - a series of XColony construction games consisting in mathematical manipulatives and educational activities (Alexe, Voica & Voica, 2014). Starting from 2D elements – planar chains of regular hexagons (Fig. 1a), one can assemble basic and complex modules (Fig. 1b, 1c).

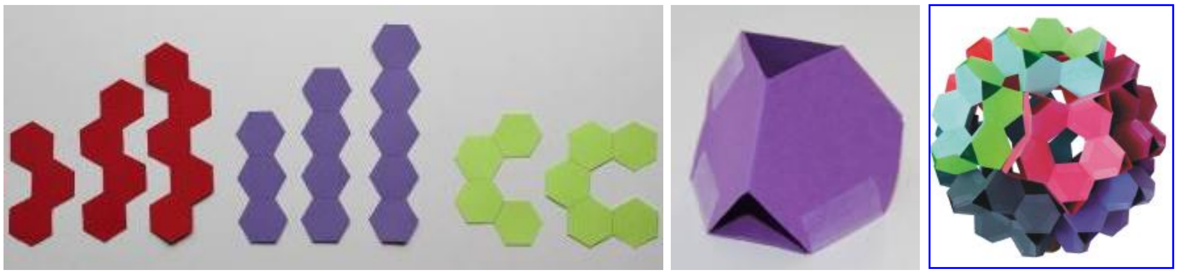


Figure 1. a. 2D Elements b. Module T c. Complex building

The modules display physical facets of hexagonal form, and co-facets (virtual faces). Two modules can be connected resulting in a more complex module; repeating operations of this kind leads to the construction of buildings of gradually increased complexity.

Sample and procedure

Our pilot study was designed as a case-control experiment in agreement with the standard practice (Vandenbroucke & Pearce, 2012) to evaluate the effect of XColony KDK activities in enhancing the geometric reasoning and creativity in 5th grade students from the *Herastrau Middle School*, Bucharest, Romania. In Romanian educational system, students are a priori grouped in "classes" which are further upheld during the entire school cycle. Classes enroll 15-30 students in the same grade, trying to maintain balanced distributions for gender. In each "class" students are taught by the same teachers; parallel classes follow the same curricular program.

Our case-control groups were selected as a priori 5th grade parallel class; due to some absentees in initial or final tests, the study was conducted with 17 students in the control group and 28 students in the target group. Each student included in the study took both the initial and the final test.

Students in the target group participated in the XColony KDK program for 8 weeks, in sessions of 1 hour per week, as part of their optional class offered through the school curriculum. During this period students in target group performed two main types of activities: they carried out tasks related to the construction of spatial structures presented in documentation and movies, and introduced by the teacher; and they were challenged to propose and ask questions like: *How else would you designate the construction you just made*? *How can you use this construction*? *What other constructions do you think are possible to be obtained*?

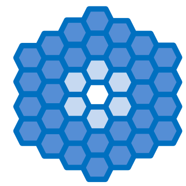
Students in the control group followed a similar curricular training program as the target group, except that the topics for the optional class were different: instead of XColony KDK they had topics in standard mathematics, including geometry elements.

Data collection

Both the target and control groups were given the same two tests: one at the beginning and one at the end of the XColony program (see Appendix). Each of the two tests consisted of 3 questions aimed to evaluate features as: spatial intelligence (Q1, Q2, Q3), geometric measurements and estimations (Q2, Q3), and composing/decomposing of geometric shapes (Q1, Q3).

For illustration, a question included in the final test is described below:

Q2 (final): *Andy would like to make an exterior frame for the picture in the image shown on the right. He wants to use hexagonal pieces of the same shape and size as the elements of the picture itself. How many pieces does he need? Explain your reasoning.*



Both tests were administered by the students' math teacher within a time range of 50 minutes. The target and control groups took the tests simultaneously. The final test was intended to be harder than the initial one, in order to increase the sensitivity of evaluation.

We expected that test data might include small amounts of variation due to local conditions: although target and control groups are coming from the same school and they are having the same curriculum, they are not sharing exactly the same teachers. For the purpose of minimizing the noise induced by such external factors the target group data was calibrated based on the control group.

Test scoring

The results of the initial and final tests were analyzed along three coordinates: comprehension (C), solution/problem solving (S), and argumentation/reasoning (A). Each question received a binary score 0 or 1 for each coordinate; the score assignment rules were defined in advance in a consistent way for both control and target groups. A *global score* (*CSA*) was defined as the average of the C, S and A scores per problem. *Delta scores* for C, S, and A were defined as the difference between final test and initial test per each score, respectively. *Av_Initial, Av_Final* and *Av_Delta* scores were computed as averages for C, S, A, and global CSA scores across all three problems per test.

Calibration

We hypothesized that the performance changes for the target group are explained through the additive effect of XColony KDK activities and the effect of external factors evaluated in the Control group: *ΔTarget.Group = XColony Effect + ΔControl.Group*, where *Δ* denotes performance changes in final vs. initial tests. Therefore the effect of XColony KDK activities was evaluated based on the score changes in the target group after *calibration with the control group.*

Calibration was performed by centering target score data to the average score of the control group, i.e., by subtracting from each target score value the average of that score across the control group, as computed at the bottom of Supplementary Table S1 from Appendix. The numerical computations were performed by using the statistical platform R-3.1.2 (http://www.r-project.org/).

RESULTS

We present the results of this study in two categories: qualitative and quantitative.

Qualitative analysis

A holistic analysis of students' answers in the initial and final test reveals several interesting aspects:

Communication. The target group improves the communication skills: in the final test, the majority of the students could provide coherent reasoning on how they got the results, and what choices have been made. The students in the control group do not show a similar development.

Argumentation/Reasoning. In the final test, students in the target group provide more variants for reasoning, vs. their initial test and to the final test of the control group. For instance, while solving Q2(final), the reasons are based on completing the missing hexagons (drawn or sketched) and then counting them; other approaches try to identify a recursion for the subsequent layers based on the rules observed for adding new pieces.

Intuition. The students in the target group show improved skills in the recognition of geometric patterns and forms. Initially, many students mistook the Hc3 element to a set of 3 hexagons, each one adjacent to the other two. Errors of this type are much fewer in the final test of the target group. Many students in the control group have difficulties with understanding the form of the element Hc5: they recognize the global pattern, but fail to understand the local structure.

Quantitative analysis

The test scores obtained by students in the control and target groups are provided in Supplementary Tables S1 and S2 from the *Appendix*. The changes in the C, S, A and global CSA test scores are depicted in Supplementary Figures S3, S4 and S5. The positive correlations between CSA global test scores and curricular math grade scores are illustrated in Supplementary Figure S6. The negative differences between scores in final vs. initial non-calibrated tests support that the final test was more difficult than the initial test.

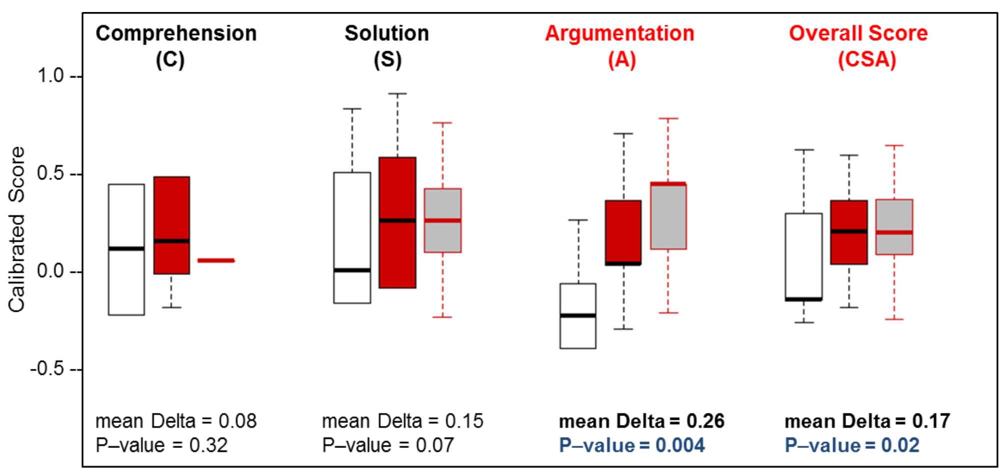


Figure 2. Comparative analysis of C, S, A, and global CSA scores changes in the final vs. initial tests based on calibrated target data. White/red boxplots present scores assigned in the initial/final tests. Grey boxplots present difference (Delta) scores for final vs. initial tests.

Calibrated target data were investigated for significant differences in C, S, A and CSA scores in the final vs. initial tests based on the 2 tailed t-test statistics (function *t.test*, R library *stats*) with the P-value cut-off 0.05. Figure 2 depicts boxplots for target group scores in the initial and final test, along with their differences (delta scores). The changes in argumentation/reasoning (26% increase) and in the global comprehension, problem solving and reasoning score (17% increase) were significant (P-value 0.004 and P-value 0.02, respectively.)

MAIN FINDINGS

Our study follows up on the research questions raised during the MCG 2014 conference, by investigating whether manipulative activities are able to encourage the development of students' creativity and to enhance spatial reasoning. The results confirm these hypotheses and show in a case-control experiment that XColony KDK activities lead to a significant increase of spatial reasoning, and of the global score for comprehension, problem solving and reasoning. More precisely, the main findings of our study are the following:

XColony activities promote understanding of geometrical concepts. The adjusted scores of the students in the target group show 26% increase in spatial reasoning, 15% in solution finding, and 7% in comprehension.

XColony activities enhance students' creativity. The results of the study indicate significant development of spatial intuition (i.e., skills required to mentally manipulate 2D and 3D objects) in students in the target group, thus confirming previous research (Alexe, Voica & Voica, 2014).

Student's score in our tests is also predictor for his/her performances in curricular math. The scores in the initial and final tests of the students in the target group are highly correlated (0.67 and 0.68, respectively) with students standard math grades in school.

CONCLUSIONS

The paper reports an empirical case-control research with forty-five 5th grade students. The study is based on the systematic use of XColony KDK for optional class activities, and on comparing the results of target and control groups in two final vs. initial tests. The results suggest that the KDK activities enhance students' creativity and visual perception, as well as mathematical thinking in a geometric context.

The KDK study contributes to the identification of comprehensive relationships between spatial education, geometric creativity and general performance in school. From a practical point of view, KDK is able to provide a flexible, personalized educational platform for fostering spatial abilities and strategic reasoning.

Future research will focus on validating the hypothesis that KDK activities improve a broad range of spatial abilities, logical reasoning and creativity in students. We plan to conduct extended case-control studies with an increased number of participants from a diversity of cultural environments, and to employ more focused tests and scoring methods as described in (Harris, Hirsh-Pasek & Newcombe, 2013). In addition, we plan to test if the effects of KDK translate in improving students' performance in various STEM disciplines (Mathematics, Physics, Chemistry, Biology), Medical Sciences, Social Sciences and Arts.

Acknowledgements. The authors gratefully thank Aurelia Grigorescu, Angela Newberry, and Wai Leng Lye for their involvement, support and feedback in this research.

Appendix. Additional details (initial and final tests, tables and pictures) about this study are available in the Appendix: www.x-colony.com/MCG9_Appendix.pdf.

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ORAL PRESENTATIONS 3.1.

Mathematical giftedness, talent and promise

Chair of the session: **Torsten Fritzlar**

ANALOGICAL-REASONING ABILITIES OF MATHEMATICALLY GIFTED CHILDREN - FIRST RESULTS OF THE VIDEO STUDY VISTAD

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Frank Förster (Technical University of Brunswick, Germany, f.foerster@tu-braunschweig.de)

***Abstract**. This paper presents initial results of the research project ViStAD (Video-Studie Analoges Denken), which is a qualitative video survey on the analogical reasoning of mathematical gifted children in the third and fourth grades. The study takes a closer look at these children's abilities to recognise analogies as characteristics of mathematical giftedness as well as at the phase in which analogical reasoning occurs during the problem solving process. 31 potentially gifted and 19 less gifted primary school children as a comparison worked on analogous problem pairs, in which the solution of the source problem could be directly transferred to the target problem, if the analogy was recognised. In this small sample, distinct differences between the test groups and the comparison groups are evident regarding their capability to recognise and use analogies autonomously as well as regarding the phase in which analogical reasoning takes place during the problem solving process.*

Key words: characteristics of mathematical giftedness, analogical reasoning, mathematically gifted primary school children, video study

THEORETICAL BACKGROUND AND FRAMEWORK

Current research suggests (Aßmus, 2013) that mathematically gifted children succeed more easily than those less gifted in recognising and using analogies between given mathematical situations. Käpnick (1998) showed that "potentially mathematically gifted third and fourth graders differ distinctly from less gifted coevals in recognising structural connections, transferring recognised structures or changing the modes of representation." (ibid. 193, q.v. 266. Translation by authors). Then again, even in the group of potentially mathematically gifted primary school children surveyed, only few demonstrated these abilities. Considering that, there is only limited empirical research on this matter and studies found are mainly based on the quantitative analysis of written documents which reveals little about children's cognitive processes (Aßmus & Förster, 2013a), we conducted the following qualitative video study. In our study, we distinguish between analogical understanding and analogical problem solving as categories of analogical reasoning. Whereas analogical understanding uses connections, modes of action, etc., to access new subject matter, analogical problem solving relies on past experience with similar problems (Hesse, 1991; s.a. English, 2004). We regard the transfer of approaches and (partial) solutions as usage of analogies, which can occur in the context of analogical understanding or analogical problem solving. In this paper, we will focus on analogical problem solving, which we consider a multi-step process involving higher-order cognitive skills. Given a new problem (target), the identification of a formerly known problem or situation (source) is required as well as a mapping between the elements and relational structure of the source problem and the target. This entails the ability to change modes of representations and therefore to abstract from the specific surface characteristics of the given situations (Novick, 1988).

RESEARCH QUESTIONS AND METHODOLOGY

In this paper, we focus on the following research questions:

1) To what extend are abilities to recognise and use analogies a characteristic of mathematical giftedness in the case of potentially gifted primary school pupils? 2) At what stage in the problem solving process does the recognition or even usage of analogies occur?

Regarding the first question, we used similar studies on less gifted primary school children as a comparison. Furthermore, we want to establish the conditions necessary for primary school children to recognise und use analogies. This also includes tracing the point of recognition (and possibly, also use) of analogies within the problem solving process.

Further research questions that interested us and which we will discuss here only peripherally include:

3) How do the different approaches chosen by the children to solve the source problem influence the recognition of analogies? 4) Is it possible to identify conditions that can facilitate, hinder or prevent the process of recognising and using analogies? In particular, how does the design of the source problem influence the recognition of analogies (distance of transfer)?

We conducted semi-structured clinical interviews (Beck & Maier, 1993) requesting our test persons to think aloud (Ericsson & Simon, 1993). To ensure, that the recognition of analogies did not take place in cooperation with other children, the probands worked individually in laboratory situations. To date, we have completed 167 interviews in our study using 12 different problems. In this paper, we will deal with the problem "triangular numbers" (test group: 15 persons; comparison group: 7) and "combinatorics V2" (test group: 16; comparison group: 12). The test groups consist of third and fourth graders considered potentially mathematically gifted due to diagnostic selection tests and/or in-process diagnostics (most of them participated in programmes for mathematically gifted students at Technical University of Brunswick or Halle University). In the comparison groups, we interviewed third and fourth graders from Halle and Wolfsburg.

One problem that occurs with studies dealing with analogies recognition by mathematically gifted children is that these children often already know methods to solve the target problem; thus, from their viewpoint, there is no need to search for an adequate source problem. On the other hand, if the target problem chosen is too complex, children of this age will quickly be incapable of coping with the situation – this holds particularly true for the children of the comparison group. We were able to observe unprovoked, random analogy recognition during the problem solving several times; however, it occurred only rarely. Such occurrences could only be documented by means of permanent "video surveillance." For this reason, many studies on transfer research (e.g. the survey in Klauer, 2011; s.a. Lobato, 1996) choose an approach, in which the test persons are first taught a solution strategy for the source problem, which then subsequently must be autonomously transferred to the target problem. Since we are not interested in investigating learning effects, we have developed analogous problem pairs, in which source problem and target problem could be solved independently of each other, however, the solution of the source problem could be directly transferred to the target problem, if the analogy was recognised.

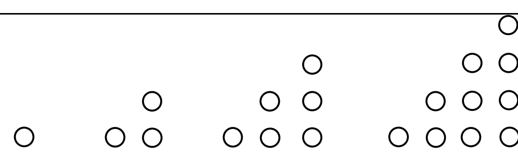
It must be emphasised that the focus of our research is on analogy recognition, not on analogies usage as in most of the other research projects. Hence, even if our test persons showed no signs of recognising or using analogies during the problem solving process, we initiated a reviewing process by asking: "Compare the problems. Which was more difficult? Why? How did you proceed? Are there similarities between the problems? …". If they still gave no indication of analogical reasoning, they were also asked graded auxiliary questions: "Look at the results of the problems. Why did you get the same result? …". This was done to find out if they could at least recognise analogies under guidance. In contrast, during the actual working on the problem no intervention on the part of the interviewer took place - regarding the probands of the comparison group, some exceptions were made when help was needed with the source problem. To enable comparability, we developed a figure sequence problem "triangular numbers" very similar to that used by Käpnick (1998) (for further and more complex problems used in the study q.v. Aßmus & Förster, 2013a).

The interviews were videotaped; the analysis was based on transcripts or process protocols including extensive excerpts of transcripts. Since the sources and reasoning of pupils' actions are not observable, but rather had to be reconstructed through interpretation (interpretative paradigm Wilson, 1973), we utilised turn-by-turn analysis of transcribed sequences (Krummheuer, 2007). The transfer of the problem solution was directly observable in some cases, but rarely the preceding recognition of an analogy. The interpretation also aims to reconstruct the phase in which the analogy recognition took place. If no observable transfer is discernible, it is questionable that an autonomous recognition of the analogy occurred at all. Therefore, our research results are based on, for the most part, the interpretation of oral and written statements, and in certain cases, on actions that were clearly recognisable.

MATHEMATICAL PROBLEMS AND FINDINGS

Triangular Numbers

Source problem



Here circles are arranged so that they make triangles. These triangular arrangements grow constantly in size according to a specific rule. How many circles will the triangular arrangement contain if it has 20 circles in the lower most row?

Target problem

What is the sum of all numbers from 1 to 20? Give reasons for your solution.

In the group of the potentially gifted children (test group), 12 out of 15 children demonstrated an autonomous analogy recognition. These children used the recognised analogy furthermore to generate or validate the result of the target problem, depending on the point of recognising the analogy. Nine children verbalised the similarities between the problems either before or while working on the target problem and solved it subsequently by transferring the result of the source problem. Three children also recognised the analogy, but only after working on both problems, so that a transfer was no longer possible. Nevertheless, they possibly used the analogy to reconfirm their results.

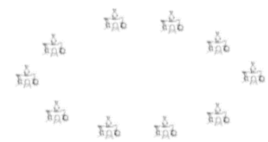
The remaining three children showed no signs of analogical reasoning and several pointed questions were necessary to provoke statements about analogies between the two problems. While one child immediately began to explain analogous structures of the problems in reply to the question "Are there similarities between the problems?", the other children needed several further questions, before they made any assumptions concerning analogies.

In contrast to the potentially mathematically gifted children, none of the seven children in the comparison group showed any indication of autonomously recognising analogies; although five children were able to identify appropriate problem structures and used them to solve the problem in the end. Statements on similarities between the problems from all seven children were only voiced after first being added with pointed questions. Three of them needed just one, two others needed several helpful questions and considerable support, while the last two showed no indication of analogy recognition at all.

Combinatorics problems (V2)

Source Problem

10 children have built a sandcastle on the beach each. Now every sandcastle has to be connected with every other sandcastle by a ditch. How many ditches must be dug?



Target Problem

At a handball tournament, each team must play against every other team. 10 teams take part at the tournament. How many matches have to be played?

With these problems as well, the majority of potentially gifted children could be observed recognising and using analogies. 13 out of 16 children transferred the result of the source problem to the target problem immediately after reading the target problem to the interviewer. They were not observed working on a new solution to the target problem thereafter. Some commented spontaneously on the analogy: "That's the same as with the sand castles." "We had it already. Because there are ten and ten should be connected with each other." Another child recognised the analogy during the reviewing process. Some rudiments were at least discernible with another other child. In one case, the test person gave no indication whatsoever of recognising the analogy.

In the comparison group, five out of twelve children transferred the result of the source problem to the target problem. One of these children initially began working on the "handball problem", but then stopped and wrote down the correct answer. The others transferred the result without working on the target problem. Out of the remaining seven children, two showed the beginnings of analogy recognition during the reviewing process; however, the other five children gave no indication of recognising the analogy.

DISCUSSION AND OUTLOOK

In this small sample, distinct differences between the test groups and the comparison groups are evident regarding their capability to recognise and use analogies autonomously as well as regarding the phase in which analogical reasoning takes place during the problem solving process. While almost all of the potentially gifted children identified the analogy between the problems autonomously, only few children in the comparison group demonstrated this ability. The potentially mathematically gifted pupils showed analogy recognition in a very early phase during the problem solving process, primarily during the activation or while building of an adequate mathematical model, so that the analogy could be used to solve the target problem. On the other hand, for most of children of the comparison groups, we observed analogy recognition only in the reviewing phase, if at all.

An analysis of the different approaches to the "triangular numbers" revealed that recognising the analogies between source and target problem is enhanced, when the source problem is initially solved using a procedure based on the summation of consecutive numbers. However, this is not in itself sufficient to produce analogy recognition. Examining the comparison group, it became apparent that in addition to structuring via summation, it was necessary to have a variable utilisation of this summation. For instance, the equality of the sums was not identified, in case the calculation was done in reverse order of the summands, despite the usage of identical numbers.

Those who recognized the analogy in the "combinatorics V2" problems normally used an arithmetical approach to the source problem. Prerequisite for this is the capability to distinguish the task structure and to develop an adequate mathematical model. This structure based working needs to be supplemented with an extensive examination of the problem, thus, establishing good preconditions for recognising analogies. On the other hand, approaches based solely on superficial characteristics of the problems (e.g. unsystematic drawing and counting) appear to have a negative effect on recognising analogies (Assmus, Förster & Fritzlar, 2014). Such approaches were only used in the comparison group.

The study's results show evidence that certain abilities for recognising and using analogies in sufficiently complex situations are characteristic of the mathematically giftedness. Due to the small database, however, these findings should be treated solely as preliminary evidence. Generalised statements cannot be made at this time.

Bauersfeld's views (2001) on the distinctiveness of highly gifted persons' "domains of subjective experiences" (Subjektive Erfahrungsbereiche, "SEB") offer another way of explaining the differences between both groups. He maintains that due to a particular sensitivity, highly gifted pupils develop more and richer SEBs, they are able to activate these SEB more easily as well as to switch between different SEBs more quickly (ibid.). Even though the children in both groups verbalised identical structures to some extent, it is possible that these structures were embedded in SEBs of varying richness, so that the mathematically gifted children could use them in a more flexible way to enable the recognition of analogies, not possible for the less gifted.

The findings in this paper apply to two particular pairs of problems. In addition, further pairs of analogical problems with a more complex mathematical structure have been tried on potentially mathematically gifted children, although, as of yet, not on a comparison group. Research confirms that it is difficult even for many of the potentially mathematically gifted children to recognize and use analogies, when problems get more complex. This occurs especially when the identified and used structures of the source problem do not coincide with the manifest structures of the target problem, as the analysis of variations of some problems showed (Aßmus & Förster, 2013b).

Further research using different numbers and solutions for source and target problems is planned.

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HIGHER ORDER THINKING IN MATHEMATICS

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Abstract. *The importance of the development of higher order thinking in mathematics is widely acknowledged. However there is no generally accepted definition of higher order thinking in mathematics. Therefore, an operational definition of higher order thinking in mathematics is needed which will also facilitate the examination of students' higher order mathematical thinking and the design of effective learning environments which promote this kind of thinking. Thus, in this paper we will try to define higher order thinking in mathematics with the use of the Integrated Thinking Model (Iowa Department of Education, 1989) and also propose tasks, based on this definition, for the assessment of higher order thinking in mathematics. We argue that higher order thinking in mathematics is a broad term which integrates basic mathematical knowledge, critical thinking, creative thinking and complex thinking processes.*

Key words: Higher order thinking, Basic knowledge, Critical thinking, Creative thinking, Complex thinking processes.

INTRODUCTION

Nowadays there is a greater demand for our schools to deliver graduates who are able to face new challenges which are the outcome of the world's growing complication, competitiveness and technological advancements (e.g., English, 2011). However, the educational systems appear not to be able to prepare students to face these challenges, since they resist change and do not utilize effectively the technological creations and challenges (Levin & Wadmany, 2006). Along the same lines, researchers in mathematics education recognise the mismatch "between low-level skills emphasized in test-driven curriculum materials and the kind of understanding and abilities that are needed for success beyond school" (Lesh & Zawojewski, 2007, p. 764). A number of researchers argued that such abilities, needed for success beyond school, may be: creativity, construction, description, explanation, prediction, and manipulation of complicated problems by designing, monitoring and communicating ideas (e.g., English, 2011; Lesh & Zawojewski, 2007). These abilities have been characterized as dimensions of higher order thinking (HOT) (e.g., Brookhard, 2010; Iowa Department of Education, 1989). Furthermore, researchers also claim that mathematics curricula with emphasis on HOT processes increase students' achievement (e.g., Boaler & Staples, 2008). Despite all these discussions, there is no generally accepted definition of HOT in mathematics which takes on board the above mentioned abilities and there is no evidence whether HOT in mathematics is something that can be accomplished by all students or simply by gifted students.

Resnick (1987) argued that "although we cannot define higher order thinking exactly, we can recognize it when it occurs" (p. 2). Therefore, a definition for HOT in mathematics should be operational, it should allow the design of assessment tasks which will also make students' thinking visible. At the same time teachers can use this operational definition to construct tasks that promote HOT in mathematics, since research results indicated that mathematics teachers have difficulty to construct tasks that promote HOT in mathematics (e.g., Thompson, 2008). However, although much work has been done in general education research, regarding the design of assessment tools for assessing higher order thinking, within mathematics education there are no valid assessment tools for assessing higher order mathematical thinking and have direct impact on instruction (Kulm, 1990).

Based on the above, the aim of this paper is to clarify the concept and define HOT in mathematics and to propose tasks for measuring it. To this end, different views of HOT are discussed based on theoretical studies from general education and mathematics education. Based on this literature review, we will propose a definition of HOT in mathematics and we will exemplify it by presenting mathematical tasks suitable for elementary school students.

THEORETICAL BACKGROUND

The concept of "Higher Order Thinking" (HOT) has been interpreted in different ways in different scientific areas. These interpretations vary significantly and are often based on different ideas (Brookhart, 2010; Lewis & Smith, 1993). The procedure to define HOT is characterized as problematic since there is no generally accepted taxonomy or typology to describe it (Resnick, 1987).

Philosophers and generally the representatives of humanities tend to believe that HOT coincides with logic and critical thinking. Based on this idea, HOT is described as a person's ability to use logic to decide what to believe and what to do (Lewis & Smith, 1993). Psychologists and generally researchers of sciences defined HOT in terms of problem solving, and the ability of individuals to solve complex problems (Lewis & Smith, 1993). Another perspective is that which defines HOT in terms of transfer of knowledge; that is the ability to use what a person learned in new situations (Brookhart, 2010). However, Lewis and Smith (1993) underlined that logic, critical thinking, problem solving and transfer of knowledge are necessary skills to understand the concept of HOT, but these concepts are not enough to give the meaning of HOT. They argued that it is a broader term, multidimensional which includes additional skills, such as creative thinking and decision making. Specifically, Lewis and Smith (1993) proposed the following definition for HOT: "HOT occurs when a person takes new information and information stored in memory and interrelated and/or rearranges and extends this information to achieve a purpose or find possible answers in perplexing situations" (p. 136). Moreover, Brookhard (2010) argued that there is a big overlap between definitions for HOT as problem solving, critical thinking and transfer of knowledge. Therefore, there is a need for a broader view of HOT. One model that presents a broader view of HOT is the Integrated Thinking Model (Iowa Department of Education, 1989). Based on this model, for someone to reach higher order thinking, a combination of content/basic knowledge, critical thinking, creative thinking and complex thinking processes is necessary. These four components are considered interrelated and dependent on each other. This model is not the only broad model of HOT. However, we chose for the purpose of our research because it offers an organised, well-defined and integrated view of HOT and at the same time we feel it is operational and can be used to observe and measure HOT. A basic argument of this model is that HOT exists in some level in all individuals from primary education (Iowa Department of Education, 1989). Moreover, this model underlines that HOT is not "a collection of separate skills but an interactive system" (Iowa Department of Education, 1989, p. 7) where there are no absolute boundaries between content/basic knowledge, critical thinking, creative thinking and complex thinking processes. In recent years researchers in mathematics education indicated and defined one or two of the above components of the model as HOT processes in mathematics. However, nobody considered all the components of the model to define HOT in mathematics. For example, a number of researchers indicated that creative thinking is part of HOT processes in mathematics in terms to create new knowledge and to provide many, different and original solutions in mathematical tasks (e.g., Silver, 1997), critical thinking is part of HOT processes in mathematics in terms of logical thinking to analyse and evaluate examples (e.g., English, 2011) and complex thinking processes is part of HOT processes in mathematics in terms of solving modeling problems (e.g., English, 2011).

Students' content/basic knowledge is the knowledge that they can retrieve directly from what they have learned (Jonassen, 2000). Critical thinking is the ability to reorganize the knowledge using the processes of analyzing, connecting, and evaluating accepted knowledge (Iowa Department of Education, 1989). Creative thinking involves "using and going beyond the accepted and reorganized knowledge to generate new knowledge" (Iowa Department of Education, 1989, p. 7). More specifically, creative knowledge is the new knowledge brought about by imagining, synthesizing and elaborating (Iowa Department of Education, 1989). Finally, complex thinking processes combine the skills and knowledge types of the other three kinds of thinking, to produce an integration of accepted, reorganized and generated knowledge. This kind of thinking includes processes such as problem solving (use systematic methods to achieve a goal), designing (create something to achieve a goal) and decision making (choose the best solution or procedure among alternatives) and it is characterized by a high level of sophistication (Iowa Department of Education, 1989).

Jonassen (2000) argued that this model is the most detailed, comprehensive and useful model for HOT. It has been used in different ways in educational research. For example, this model has been used to design observation tools for teaching (e.g., Iowa Department of Education, 1989), to define evaluation criteria of potential technological tools (e.g., Jonassen, 2000) as well as for other purposes.

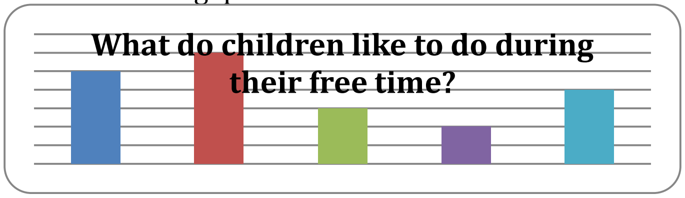
HOT in mathematics has been defined and measured in several ways. Many researchers define HOT in mathematics, as the ability to solve non routine problems (e.g., NCTM, 1989). However, there are also definitions where HOT in mathematics is viewed as a broader term than simply the ability to solve non routine problems. For example, Wilson (1971) claims that HOT in mathematics also includes the ability to solve routine problems, make comparisons, analyze data, recognize patterns, discover relationships, construct and criticize proof and find generalizations. Similarly, Foong (2000) argued that HOT in mathematics requires from students to make predictions, communicate their ideas, choose their solution path and evaluate their solution. Therefore, there is a need to have a comprehensive definition for HOT in mathematics, which will be widely accepted.

THE PROPOSED MODEL FOR HOT IN MATHEMATICS

Based on the Integrated Thinking Model (Iowa Department of Education, 1989), we argue that HOT in mathematics is a multicomponent concept which all students from elementary school can develop. For someone to achieve HOT in mathematics it is necessary to integrate basic mathematical knowledge, critical thinking in mathematics, creative thinking in mathematics and complex thinking processes in mathematics. What follows is a brief explanation of each component accompanied by an example for sixth grade students.

Basic mathematical knowledge is the mathematical knowledge and understanding which constitutes the basis for an individual to extend his/her mathematical thinking. An indicative example, for sixth grade students, is illustrated in Figure 1. This task requires from students to read and understand the information given by the graph in order to answer some questions. Specifically, it asks students to find and compare information from the graph.

Use the graph to answer the following questions.



(a) How many children like to read books in their free time?, (b) What do most children like to do in their free time?, (c) How many more children like to play computer games than read books in their free time?

Figure 1: Task that requires from students to use basic mathematical knowledge.

Critical thinking in mathematics refers to the ability to analyze, connect and evaluate information based on criteria or logic in order to take a decision or solve a problem. For example, sixth grade students may be asked to find among four data bar graphs the one which presents the data of a given table (see Butterworth, & Thwaites, 2013). We argue that such task requires the use of critical thinking, since students need to analyze both the data graph and tables, identify similarities between them (connection) and evaluate which graphs show the data presented on the table. This task requires from students not to simply read information from the graph, but also to reorganized these information to give an answer.

Creative thinking in mathematics can be described as the act of "integration of existing knowledge with mathematical intuition, imagination, and inspiration, resulting in a mathematically accepted solution" (Levav-Waynberg & Leikin, 2009, p. 778). Thus, creative thinking in mathematics surfaces when students use imagination to synthesize in an original way information or produce many and different solutions through elaboration. Such a task is presented in Figure 2. In this task sixth grade students are asked to write as many different questions as possible which can be answered based on the information presented by the graph. Students in this task are expected to use the procedure of synthesizing to write a question, imagination to present more than one questions and elaboration to present different types of questions. This task is a problem posing task, a characteristic of creative activity (e.g., Silver, 1997).

Write as many different questions as possible which can be answered by the information provided by the graph. Try to write questions which no one else can think of.

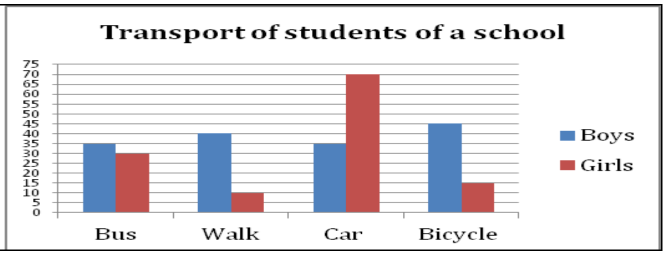


Figure 2: Mathematical creativity task.

Complex thinking processes in mathematics are those processes required by individuals in order to respond to complex problems. These include a combination of basic mathematical knowledge, critical thinking and creative thinking. An indicative example is presented in Figure 3 for sixth grade students. In this task students should read and understand the information of the graph (e.g., the meaning of different colors and axes) (basic knowledge). Then students are expected to write down the number of students of the two groups for every range of scores (horizontal axis), to compare these numbers in order to justify that there is a need to calculate a mean of grades of each group of students (critical thinking processes). Following, students are expected to realize that to calculate the mean they need a score for every student so that Group A students will have better test results than Group B students (e.g., they should realise that this may happen if Group A students take the maximum score within each range of values and Group B students take the minimum score within each range). Finally, students should synthesize a mathematical argument (creative thinking processes). A similar task was proposed by Aizikovitsh-Udi, Kuntze and Clarke (2014) as suitable to promote HOT processes in mathematics (statistical thinking and critical thinking).

Based on the following graph, the teacher claims that students of group B had better results in the mathematics test than group A students. Group A students disagree. Write a mathematical argument which will convince the teacher that Group A students had better test results that Group B students. (Adopted from OECD, 2006)

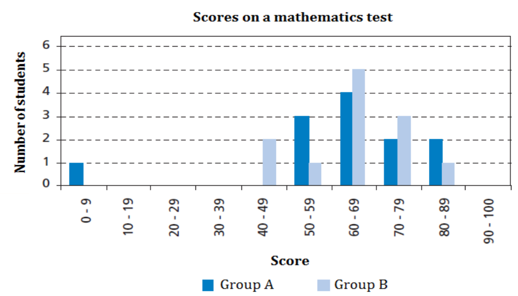


Figure 3: Mathematical task that requires students use complex thinking processes.

The components of HOT described above: basic mathematical knowledge, critical thinking in mathematics, creative thinking in mathematics and complex thinking processes in mathematics are interrelated. Specifically, without basic mathematical knowledge students cannot develop critical thinking, creative thinking and complex thinking processes. Moreover, the use of critical, creative and complex thinking processes in mathematics can enhance mathematical knowledge and understanding. Critical thinking processes can be used to produce multiple, different and original solutions (creativity thinking). Additionally, the creative thinking processes enhance students' development of logic arguments (critical thinking).

CONCLUSIONS

In this paper we proposed an operational definition for HOT in mathematics based on the Integrated Thinking Model (Iowa Department of Education, 1989). Our future plan is to provide an empirical verification of this proposed HOT definition by examining elementary school students. Additionally, we will design and investigate the effect of different technologically based learning environments for developing HOT in mathematics.

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EXAMINING COVERT IMPEDIMENTS TO INCLUSIVE EDUCATION FOR THE MATHEMATICALLY GIFTED LEARNERS IN SOUTH AFRICA

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Abstract. *This paper argues that there are covert impediments in the South African Education System that pose a threat to the inclusion of the mathematically gifted learners. The concern arises from empirical evidence showing that the country had made significant strides in widening participation of the school going children yet there was no meaningful learning going on in the classrooms. This was more detrimental to meeting the needs of the mathematically gifted learners in such mainstream classroom. Five of these impediments linked to ambiguity in policy pronouncements; lack of epistemic access; poor quality of teaching; curriculum rigidity and lack of support for the mathematically gifted learners are identified and examined. The paper concludes by suggesting that truly inclusive education needs to provide all children with an equal opportunity to learn and to develop to their full potential.*

Key words: inclusive education, gifted education, mathematically gifted, epistemic access, barriers to learning

INTRODUCTION

Despite the fact that gifted education falls under the umbrella of special education as part of the continuum of exceptionalities, implementation of inclusive education most often ignores the unique needs of learners who are gifted (Moltzen, 2006). This has prompted critics to argue that the very educational system, whose purpose is to prepare our children for the future, has categorically been destroying the future of our gifted children. In South Africa the introduction of inclusive education was spelt out in Education White Paper No. 6 on Special Needs Education whose primary aim was to create an educational system where all learners would be educated in mainstream schools and classrooms (Department of Basic Education, 2010). However, observations are that South Africa has made significant strides in addressing physical access which has not translated into meaningful learning (Motala & Dieltiens, 2010). The questions now at stake are: How is inclusive education conceptualized and operationalized within the South African education system? How does this affect learners who are considered mathematically gifted?

WHY THE SPECIFIC CONCERN WITH MATHEMATICALLY GIFTED

This concern for the mathematically gifted learners follows Jha's (2002) observation that despite rhetoric of inclusive education, policies in both developed and developing countries continue to ensure that vulnerable and disadvantaged groups are often excluded from forms of education regarded as most valuable, and from gaining qualifications that can be exchanged for good employment, income and security. This paper argues that mathematically gifted learners fall into this category of vulnerable groups who are excluded from forms of education regarded as valuable. There is a global discourse that positions mathematical competence as the key to the welfare of a nation in a global economy. Mathematics education is therefore viewed as the school subject that can save excluded children from their lack of a future hence it has been argued that who gets to learn mathematics and the nature of the mathematics that is learned are therefore matters of consequence (Persson, 2010). Within these debates, international studies warn that the student most neglected/excluded in terms of realizing full potential was the talented student of mathematics. Particularly at risk of this exclusion are the gifted children from economically disadvantaged backgrounds as they cannot afford to receive appropriate education elsewhere yet they do not have protection under national statute for a free and appropriate public education (Kearney, 1996).

Objectives of the paper

The debates on access to education suggest that most school systems are confronting overt and covert barriers to school inclusion (Jha, 2002). In attempting to address these barriers many developing countries tend to focus on increasing the participation of students (overt barriers) but ignoring all the other ways (covert barriers) in which participation for any student may be impeded or enhanced (Macedo, 2013). These covert impediments to learning need to be investigated as they leave mathematically gifted learners vulnerable to exclusion and learning breakdown. In South Africa there is limited research done on how gifted learners are catered for in the regular classroom. Given this gap; this paper analyses five covert impediments, which are associated with ambiguity in policy pronouncements, lack of epistemic access, poor quality of teaching, curriculum rigidity and lack of support for the mathematically gifted learners.

CONCEPTUALIZING INCLUSIVE EDUCATION

Given that inclusive education is central to this paper it is important that I clarify how it is conceptualized in this paper. Inclusion is a philosophy combining many theories but has its roots in social justice, and the civil rights movements of the 60's. Inclusive education is in essence a socially complex process and as such various definitions are possible leading to various positions being adopted. In this paper I take the definition given in the UNESCO, 1994, Framework for Action on Special Needs Education, (p.6) which states that a school should,

> …accommodate all children regardless of their physical, intellectual, social, linguistic or other conditions. This should include disabled and gifted children, street and working children, children from remote or nomadic populations, children from linguistic, ethnic, or cultural minorities and children from other disadvantaged or marginalized areas and groups".

This was the original conceptualization of inclusive education, a system that accommodates all, but at implementation stage it is sometimes politically expedient for educators and policy formulators to manipulate the term to suit whatever practice they happen to be currently engaged in. The approach has therefore been different in many developing countries where a large proportion of children are still out of school. Historically, the inclusive education movement in many developing countries was focused primarily on people with disabilities and learning difficulties. However, critics have continued to question the usefulness of an approach to inclusion that, in attempting to increase the participation of students, focuses on the 'disabled' or 'special needs' part of them and ignores all the other ways in which participation for any student may be impeded or enhanced (Macedo, 2013). This has forced some researchers to reflect back on the original conceptualisation of inclusive education which accommodated all. According to Berlach & Chambers' (2011) recent philosophical underpinnings for inclusive education include: availability of opportunity, acceptance of disability and/or disadvantage, superior ability and diversity; and an absence of bias, prejudice and inequality. Ofsted's (2012) guidance was that educational inclusion was 'more than a concern with one group of pupils … It is about equal opportunities for all children and young people whatever their age, gender, ethnicity, attainment or background' (p. 4). It is this view of inclusion that this paper takes.

Ambiguity in policy as a hidden barrier

Given this position, the first concern of this paper is how inclusive education is conceptualised in South Africa and the implication this has on the education of the mathematically gifted learners. Jansen, (2001) criticized the policy process generally arguing that some South African policies are enacted for their political symbolism rather than their practicality; thus vague polities often get passed. With specific reference to the implementation of an inclusive education policy in South Africa it is argued that this policy is at an apparent standstill due to a number of factors. One of the observations made is that the policy remains purely symbolic because it lacks specific instructions for implementation by stakeholders and there is no commitment on the part of education officials to spearhead its implementation (Oswald & de Villiers, 2013). Another concern is that in the policy for inclusive education the gifted learner is mentioned (in passing) as one category of exceptionality that should become the central part of the organisation, planning and teaching at school but this is not foregrounded within the guidelines for mainstream classroom activities (Kokot, 2011). The policy is also silent on how this has to be implemented hence educators manipulate the term to suit whatever practice they were implementing. Another observation is that right through the policy documents inclusion has been proffered through a disability lens hence researchers indict policy-makers for propagating a 'narrow view' of a democratic and inclusive education that excludes gifted learners (Kokot, 2011). This tends to work against the broader sense of inclusive education which accommodates the gifted learners as well.

Lack of epistemic access as a hidden barrier

Traditionally, debates on access to education have focused primarily on 'physical access' a term which emphasizes addressing the barriers that limit the ability of learners to physically locate themselves in an institution of learning. In South Africa physical (instead of quality) access to education was prioritized post democracy given the political history of the country (du Plooy & Zilindile, 2014). However, the assumption that simply 'getting through the school gates' (widening access) was an important but insufficient as a measure of meaningful access. Current discussions and concerns around access are no longer on increasing the enrolment numbers (expansion), but focus on the removal of structural and institutional barriers to progression and increasing the numbers of under-represented groups in the highly selective mathematics and science programmes. In this regard Morrow (2009) distinguished between two forms of access i.e. formal access and epistemic access. Formal access concerns registration at an educational institution where emphasis is on entry to a physical institution. Epistemic access on the other hand is what a curriculum should entitle the students to – it refers to how teachers shape and guide inquiry in the the discovery of truth (Marrow, 2009). If epistemic access, as Morrow suggests, refers to 'how we shape and guide enquiry… in the discovery of truth', it seems unarguable that it must be the primary, if not the basis of any system of education.

In highlighting this lack of epistemic access, Motala & Dieltiens (2010) warned that while statistical results show a South Africa which is near to achieving the 'Education For All' goals (formal access), this physical access has not translated to meaningful (epistemic) access as the learners are yet to reach the required level of achievement and competency. This observation suggests that if gifted learners are physically at school in the regular classrooms (widening participation), but are not achieving at least as well as they ought to, then it cannot be said that such children have meaningful access to education at all. So unless questions about quality of knowledge are raised concerning the mathematically gifted learners, then widening participation can lead to little more than the 'warehousing' of these young people – a term used to describe how the majority of the unemployed school leavers in England in the 80's left with little, if any, knowledge than they had when they began their training programmes (Coffield, 2000). In summary these observations suggest that access to education for the mathematically gifted learners should be more than mere physical access, since it should be reflected in educational outcomes or post-enrolment experiences.

Lack of support as a hidden barrier

For more than a decade in South Africa, critics have consistently argued that far too many of the mathematically gifted students currently do not stand even the remotest chance of achieving near their potential because they are not receiving adequate support within mainstream classrooms (Kokot, 2011; Xolo, 2007; Oswald & de Villiers, 2013). There are a number of problems that have been raised concerning the movement toward full inclusion by teachers and others in the field. One of such concerns is that as a consequence of recent inclusion initiatives, the general classroom is becoming a place where teachers are expected to meet the needs of a wide range of students and many teachers interviewed by Oswald & de Villiers (2013) clearly stated that they had been trained to address the needs of the learner who struggled rather than the learner who was gifted. This suggests that the education of the mathematically gifted in the mainstream would suffer as a result of inclusive education because the teachers devote more time to the learners with learning disabilities. This paper therefore argues that the mathematically gifted learner has no support in the South African classroom.

Curriculum rigidity as a hidden barrier

Inspired by the goal of leaving no child behind in basic skills, education in South Africa underwent reform to make it easier for the average student which has been viewed as the 'dumbing-down' of the curriculum (Benbow & Stanley, 1996). The vast majority of highly gifted children then got caught in an 'age-grade lockstep' which routinely offered such children academic work five to eight years or more below their intellectual level (Gross, 1993). Such a standards-driven curriculum, especially in the public school, is not preparing gifted children to be innovators at the highest technical levels—the levels that will pay off most in our modern, high-tech, science-driven, global economy. According to Benbow & Stanley, (1996) the situation is analogous to requiring the talented pole-vaulter to practice only on the 6 foot-bar in preparation for the Olympic trials. The end result is that highly able students are now exposed systematically to less challenging knowledge hence they are given less chances to develop expertise. Admittedly, all children need to learn to read, but the future of developing countries like South Africa does not depend on how many children, rich or poor, gain basic skills. It depends on how many children will be able to handle the complex and technical languages, symbol systems, and practices needed for success at the highest levels of the value chain—the "languages" of higher order mathematics, complex technologies, and the hard sciences (Benbow & Stanley, 1996). The crux of the argument put forth in this paper is that if gifted students are to have equal access to knowledge that is 'powerful' and necessary in the transition to higher education and beyond, then inclusive schools should follow flexible curricula that would respond to the diverse needs of such children.

Poor quality of teaching as a hidden barrier

Teachers play a central role in promoting 'inclusive education'. Paradoxically, as Mittler (2000) notes, they represent the single greatest obstacle to inclusive education because of a number of factors including their levels of knowledge, perceptions and attitudes. Research shows overwhelmingly that good teaching is vital for better results hence according to Henning (2012) a country cannot claim social justice in education if teachers do not know their subjects, and if they do not know how the children and youth that they teach learn these subjects. In South Africa findings from inclusive education studies by Lumadi (2013) and Oswald & de Villiers (2013) show that teachers were not trained to deal with the needs of the gifted so they could not teach them effectively.

Besides this specialized knowledge for teaching gifted learners, in South Africa teachers are generally accused of not having sufficient content knowledge and this also has implications for adopting a social justice approach to education (du Plooy & Zilindile, 2014). From a study on teacher quantity and quality by the Centre for Development and Enterprise, the results show that there is a serious shortage of qualified teachers of mathematics and many of those who are teaching this subject are not qualified to do so (McCarthy & Bernstein, 2011). This shortage is further exacerbated by the fact that the existing teachers are not being optimally utilized. For example, statistics show that about ten thousand teachers migrated to the United Kingdom alone in July 1997 and July 2006. While this figure was not broken down by subjects, it is common knowledge that mathematics and science teachers are in highest demand worldwide. Teachers who emigrate also tend to be among the most talented, thus depleting the country's already inadequate pool of good teachers. A further complication is that as many as 44% of those who are qualified to teach scarce subjects such as mathematics were actually teaching other, non-scarce subjects (McCarthy & Bernstein, 2011). All this suggests that teacher quality in South Africa is weak to support mathematically gifted learners to develop to their full potential.

Conclusion

This paper has argued that the inclusive education in South Africa is proffered through the disability lens, the policy does not foreground the education of the gifted learners, teachers are not trained to meet the needs of gifted learners hence there is no epistemic access for the mathematically gifted learners. The current state of gifted education in South Africa is therefore perceived as not encouraging. In such a system not only are talents lost and bad work habits reinforced, but compelling students by law to attend school, and then limiting academic challenge for some students while providing it for others, is inconsistent with the true meaning of inclusive education. Truly inclusive education needs to provide all children with an equal opportunity to learn and to develop to their full potential. If inclusionary classrooms in South Africa are committed to serving all students, they must therefore choose to include both physically and philosophically, even the most extremely gifted children as well as children with the most severe disabilities.

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"THE WEEK OF MATHEMATICS" – A PROJECT WITH CREATIVE APPROACHES

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Abstract. *The Week of Mathematics project has been organised for the last seven years in The International Computer High School of Bucharest, Romania. This project encourages students to engage in learning mathematics in a non-formal and potentially creative context. Our poster presentation aims to present the design of the project, the activities proposed to students, and some artefacts generated in project activities.*

Key words: educational project, creative behaviour.

INTRODUCTION

A brief presentation

In the last seven years, *The International Computer High School* in Bucharest, Romania has developed the project named "The Week of Mathematics".

More precisely, during a week in the second semester of the school year, students and teachers are involved in various mathematical activities. The activities could be the following: making and presenting drawings; watching pictures on various mathematical subjects; playing mathematical games, etc.

The aims of this project were:

- to introduce students to a different way of learning mathematics;
- to develop students' creativity and imagination.

The Week of Mathematics' activities also suppose some assignments proposed to the student.

In organising these activities, we started from the following hypothesis: involving students in uncommon activities could help them improve their mathematical and communicational abilities.

Two examples

Below, we briefly present two of the most interesting activities of the project.

In the first one, we asked the students to draw a poster (in A3 format), on various unusual topics, as for example: "*The Mathematical Town*", "*The Country of Radicals*", "*The Empire of Angles*", "*A Day in the Life of a Prime Number*", etc. We gave students only general indications; every participant had the possibility to construct and present his own proposal.

Another activity consisted in building some geometric bodies (as prisms, pyramids, spheres, etc.) out of cardboard or other materials. The bodies were later on covered with funny dialogues connected to mathematical definitions.

SOME RESULTS AND CONCLUSIONS

All the activities developed in this project are specially designed to develop a creative context for the students' activities, as presented in Stănciulescu, Belous, and Moraru (2008).

Based on the results obtained in the past years, we have tried to assist and improve the Week of Mathematics' activities, by:

- Supporting students with materials;
- Facilitating students' access to the projects from previous years;
- Creating a challenging environment by granting awards and dividing the activities in several sections.

We observed that the project activities are relaxing activities in which the rigidity of the Mathematics taught in class is replaced with curiosity, freedom in communication and creative behaviour (as described in Roco, 2001).

We think that the project developed a creative environment for our students, who thus have the opportunity to present their ideas and their visions on some mathematical notions acquired during teaching classes.

The results of the project were some artifacts made by students. As pointed in literature, artifacts become instruments of students' mathematical activity and assist the construction of mathematical meanings (e.g. Maschietto & Bussi, 2009; Maschietto & Trouche, 2010).

We set on to continue and to develop the project activities, by involving a larger number of students in project activities.

PRESENTATION OF THE POSTER

The poster shows the activities carried out during "The Week of Mathematics", which made students express their creative side. We will also present some artifacts made by students during the project, and students' feedback on their activity.

This study could be a lead-in new activities and strategies meant to continue our achievements.

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ORAL PRESENTATIONS 3.2.

Challenges in the teaching practice

Chair of the session: **Linda Jensen Sheffield**

A METHOD OF TEACHING MATHEMATICS IN KOREA SCIENCE ACADEMY

Choi Jong Sool
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Abstract. *Even among Scientifically gifted students, there are some students who do not like studying mathematics or want to avoid studying mathematics if possible. How can we help them? We want to find a way help them to have interest in mathematics while they participate actively in mathematics classroom activity and do mathematics. In this paper, we show a method of teaching, the discussion and presentation oriented class, we used in Korea Science Academy of KAIST which is a special school for scientifically or mathematically gifted students in South Korea. We also show how students reacted toward this method of teaching.*

Key words: discussion, presentation, the discussion and presentation oriented class, scientifically gifted students

INTRODUCTION

In Korea Science Academy, the most mathematics courses have been given in the way of lecture-centered class which means lecturing uses more than half of class hour. Even in the lecture-centered class, some might have classroom activities such as discussion, presentation and problem solving, but it spent less than half of class hour

Although Korea Science Academy is a special school for scientifically gifted students, there are some students who do not like studying mathematics or want to avoid studying mathematics if possible. How can we help them? We want to find a way help them to have interest in mathematics while they participate actively in mathematics classroom activity and do mathematics. But this method should also be helpful for those who love studying mathematics.

For a few years, we used discussion and presentation for conceptual understanding toward Definition and Theorem in my Calculus3 class. We gave them sets of discussion problems in a way that they could derive new definition and new theorem. We could notice that this help students to change their cognition of the definition and theorem. Although some really liked this method, some were complaining that it was hard for them. We want to make it a little bit easier and increase students' activities.

We now try to remove lecturing in my Calculus 2 classes. For this purpose, we make a set of problems for my each class hour and let students to solve the set of problems by themselves or discussion with a group of students, and then assign a few selected students to present their solution. After that we discuss the presented solutions.

This is the report how my students reacted toward this teaching method.

RESEARCH METHOD

Participants

In this research, 32 students (age 16-18) who enrolled for my two Calculus-2 classes of Korea Science Academy participated in the classroom experiment of the research for whole Fall semester 2013.

30 of them were Korean and 2 of them were foreigner. All of them were 11th-graded. There are 5 required mathematics courses in Korea Science Academy and Calculus-2 is 4th one. On each course, they have two choices, either honor course or regular course. In the Fall 2013 there were six Honor Calculus-2 Classes (87 enrolled) and three calculus-2 classes (52 enrolled)

Language

We used English as the official language in the classroom. (It was required as a school policy) None of students were native speakers in English.

Overview

We tried to eliminate every sort of allusion about research in the classroom. So we chose problems in a way that we would use them in our lecture. We had no pre survey, but only a post survey. There is one comparison class in calculus 2. So we gave the same Midterm, Final, and two quizzes. We kept classroom observation for whole semester.

Classroom Experiment

Each class proceeded in the following order:

- Hand out problems in the beginning of class
- Give about 5 - 10 minutes for everyone to try to solve problems
- Assign each problem to a selected student.
- Selected students try to solve the assign problem and then present their solution while other students try to solve all problems.
- Discuss the presented solution with whole students.

Students were asked to read textbook before they come to the classroom. We selected 3 – 6 students simultaneously and they present their solutions at the same time so that we could save our time. When we discuss student's solutions, we talked not just right or wrong, but about good things and weak things, and how we can improve weak things. Unfortunately, none of students were native speaker in English. So students couldn't participate as much as they want in the discussion, Rather I had to speak more than I wanted. The following is one of the problem set we used

(Calculus II)		Instructor	JongSool Choi
Section	**11.10 Taylor and Maclaurin Series**		
Objective	To learn Taylor polynomials and Taylor's theorem for $a = 0$, Maclaurin series for important functions, and the binomial series formula for $(1 + r)^r$ $(r \in R)$ and its usefulness in finding certain Maclaurin series.		

Contents

Ex1) Recall that $L(x) = f'(0)x + f(0)$ is the linear approximation (or a linear model) for f near $x = o$. Try to find a quadratic approximation for cosx near $x = 0$. Discuss how we can find a better approximation.

Ex2) Suppose that f has a power series representation at a, that is

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n \quad |x-a| < R$$

Find c_n in terms of f and explain related terminologies.

Ex3) (a) Show that the Maclaurin series of sinx is the following;

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

(b) Is $1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n+1)!}$ the Maclaurin series for $f(x) = \frac{\sin x}{x}$? Explain why if it is. If it is not, find the function whose Maclaurin series is the given one.

Ex4) State the theorems related to the equality between a given function and its Taylor series.

Ex5) Find the Maclaurin series for $\ln(1 + x)$, and show that it is equal to its Maclaurin series.

Ex6) Show that sinx is equal to its Maclaurin series.

Ex7) Estimate $\int_0^1 \sin(x^3)\, dx$ using a power series

Ex8) Find the Maclaurin series of the following function;

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Is f(x) equal to its Maclaurin series ?

Ex9) Compute $\lim_{x \to 0} \frac{\sin x - x}{x^3}$ using power series.

Ex10) State the binomial series Theorem

Ex11) Find the binomial series for $\frac{1}{1+x}$, and compare with its power series representation obtained by using the geometric series.

Ex12) Compute $\frac{1}{(1-x)^3}$ by binomial series, and check the answer by differentiating $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ twice.

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Assessment

We evaluated students through Homework and Attitude(10%), Quizzes(20%), Midterm and Final(30% for each), and Attendance(10%)

RESEARCH RESULT AND ANALYSIS

We analyzed the result of post survey and personal observation in this section. Although the official language in the classroom was English, the survey conducted in Korean which is the native language for most of students. For two foreign students, appointed buddies helped them to answer questions in the survey. On most of major questions, we asked "Do you agree to …" and they have five choices in answer; strongly agree, agree, so so, disagree, and strongly disagree.

Question 1

Do you agree to "I liked to present my solution in the classroom"?

85% of students gave non-negative answer for this question, positive was 44%. 67% of answers about "why" chose a reason as "teacher corrected my mistakes in the presentation and this process helped me to adjust my personal wrong habit", and 13% of them chose a reason as "I were proud to be able to present correct answer"

Question 2

Do you agree to "I'd like to extend the discussion time if the official language of the classroom were Korean"?

All answers for this question were positive. It shows everyone felt that they couldn't have enough discussion due to the language problem. Also 88% of them expressed the desire to conduct the discussion on his (or her) solution if the official language were Korean and 40% of them showed the strong desire to do so.

Question 3

Do you agree to "I like Mathematics Class with Discussion and Presentation"?

93% of students answered non-negatively, and 52% of students gave positive answer. 35% of them chose a reason as "more active atmosphere", 30% of them as "less boring" and 20% of them as "I like to solve problems by myself".

Question 4

Do you agree to "I'd like to take Mathematics Class with Discussion and Presentation again"?

93% of students answered non-negatively, and 56% of students gave positive answer.

Question 5

Do you agree to "I did preview study enough to follow classroom activity"?

44% of students answered negatively, and only 22% of students respond positively. Among the negative response, 59% of them mentioned a reason as "I had too many other things to do to study mathematics enough", 24% of them chose a reason as "I didn't like to study in advance". But 70% of students said they increased the amount of study in advance in comparison with lecture-oriented class they had before, and only 7% of students said they studied less than before in preview study.

Question 6

Do you agree to "When I found a difficult material to understand in preview study, I checked it to ask teacher about it"?

33% of students answered negatively, and only 19% of students responded positively.

Question 7

Do you agree to "I have increased my review study for this math course"?

88% of students gave non-negative answer, and 54% of students answered positively. In the question how to get over the difficulty they found in the classroom activity, 70% of them chose "I ask it to my friends", 15% of them "I ask it to the teacher" and the rest "I try to get over it by myself".

Question 8

Do you agree to "After I took this class, my studying habit is changed"?

This question was asked to answer "Yes" or "No" and then "How?" 60% of students answered "Yes" and 40% of them said they increased the amount of preview study, 20% of them said they increased study time for mathematics, also there were answers as "I solve math problems with more thinking" and "I solve math problems more systematically".

Question 9

Do you agree to "I'd like to receive the worksheet right before we use it"?

63% of students responded positively and 22% of them did negatively. But in the question how we can improve, 37% of students responded. 80% of respond said they want to receive the worksheet in the class right before the class they will use it. 20% of them said they want to receive it a week before.

Question 10

Do you agree to "I think that problems in the worksheets were proper for me to understand and study the materials in the text"?

No negative answer appeared in this question and 89% were positive.

Question 11

Do you agree to "I liked the way teacher chose presenter"?

96% of students respond non-negatively, and 57% of them respond positively.

CONCLUSION

After a few weeks past from the beginning of the semester, one of my students approached me and expressed his wish to have lecture-oriented class. He explained that the amount of learning from classroom activities was less than he expected and he also mentioned that he could understand things better that he didn't in his preview study if it were lecture-oriented class. Since I already observed that his preview study had been better than most of students, but his participation in the discussion were rare and he didn't pay real attention to discussions, I recommended him to ask in the discussion time the points he didn't understand in the preview study. After that conversation, he didn't express any dissatisfaction anymore. That he took my calculus 3 class in the next semester showed his dissatisfaction about the discussion and presentation oriented class disappeared. As we see in his case and post survey, the overall reaction about this new type were not negative. More than half were positive. Major reasons students mentioned were teacher's consistent correction about their wrong reasoning, from which they could correct their personal wrong habits, more active atmosphere, and less boring. What change do we see in student's interesting and attitude toward mathematics? Although not everyone, some students said that they recognized the importance of self-study and increased the amount of study in mathematics. After half of semester passed, one of students expressed personally to me that she began to like Mathematics first time in her life.

How did this new method of teaching affect student's problem-solving ability? We could compare student's test results with the result of comparison class. But I think more cause than we have from this research could affect in the answer of the question. So we want to postpone concluding it.

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PROSPECTIVE TEACHERS' VIEWS OF CREATIVITY IN SCHOOL MATHEMATICS

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Abstract. *The aim of the study is to explore prospective teachers' view of creativity with respect to characteristic and practices of a creative teacher, characteristic of a creative student. Data collected through interview with 2 prospective mathematics teachers. Then, we analysed data by using descriptive analysis. We determined that the prospective mathematics teachers' views on creativity are limited to the classroom activities prepared by the teachers and the students' approaches to problem solving. It is concluded that the most effective determinant for this is due to cultural and contextual factors.*

Key words: Prospective teachers, Creativity, high school

INTRODUCTION

Creativity is one of the essential keystone in mathematics (Sriraman, 2004) due to the findings revealing that teacher's awareness and knowledge of creativity have a substantial significance in order to foster students' creativity (Runco & Johnson, 2002). A mathematical task could be creative by itself, but it does not make any sense if the teacher could not promote creativity in the classroom (Panaoura & Panaoura, 2014). In that regard teacher has an important role in activating creativity in the classroom (Levenson, 2013) but teachers usually think that logic has got the utmost importance in learning mathematics and ignore the power of creativity (Pehkonen, 1997). Many researches (Bolden, Harries, & Newton 2010; Shriki, 2010) confirmed the situation that teacher's knowledge on creativity are limited and narrow. It is observed that pre-service teachers' views on creativity based on the idea of 'teaching creatively' rather than 'teaching for creativity' (Bolden, et.al., 2010).

Considering the importance of fostering creativity, studies on the teachers' view on the being creativity in the context of teaching are limited in number. Despite the importance of creativity, in the Turkish curriculum, mathematical creativity in the teaching is underestimated in high school level (MoNE, 2013). Therefore, the present study reveals prospective teachers' view of creativity with respect to (1) characteristic and practices of a creative teacher, (2) characteristic of a creative student in the Turkish context.

THEORATICAL FRAMEWORK

Creativity is a multi-faced concept that many scholars identifies from various aspects (Mann 2006; Pehkonen, 1997; Sriraman 2005). For this reason, different definitions (Leikin, Subotnik, Pitta-Pantazi, Singer, & Pelczer, 2013), various approaches and interpretations have been presented about creativity and mathematical creativity in particular (Shriki, 2010). A common definition on creativity is based on Torrance (1974); he defines creativity with the basis of four main components fluency, flexibility, originality and elaboration. Creativity has a dynamic feature that students can develop if teachers provide them appropriate learning environments (Leikin, 2009). Lev-Zamir & Leikin (2011) investigated that teachers' conceptions of mathematical creativity is consisting of teacher-directed and student-directed conceptions. Teacher-directed creativity relate to teachers' actions in the context of creative teaching like generating original tasks, using different model and manipulative. Student-directed creativity relates to opportunities to foster students' creativity.

In the literature, Shriki (2010) argues that teachers have insufficient knowledge about creativity. Bolden, et. al., (2010) analyzed written questionnaires and semi-structures interview with prospective elementary school teachers about their conceptions of creativity, and showed that these conceptions were narrow and related with their own actions. In another study, teachers describe creative environment that comprise open-ended activities and non-routine problems that give students freedom to apply imaginative ideas and find novel methods or solutions (Shriki, 2008). Sheffield (2006) argued that a task that enables further questioning can foster mathematical creativity. Furthermore, teachers think that cooperation with classmates activate creativity (Fleith, 2000).

METHODOLOGY

The participants of this study were selected amongst the prospective teachers studying their last year in the mathematics teaching program in Turkey. This study aims to reflect the views of the prospective teachers, who did not receive any education on creativity throughout this program, about creativity within the context of education and training.

The prospective teachers gained experience about lesson plans and planning activities in their "teaching methods on mathematics education" lesson, which they studied for two semesters. Furthermore, they made observations and carried out various studies in classes in high schools within the scope of their "school experience" lesson. One of these studies is implementing one of the activities, which they planned earlier, in the classroom. Briefly stated, these include the prospective teachers' experiences on education, lesson plans, activities and the studies they carried out within the scope of school experience lesson.

In this study, two prospective teachers were interviewed. The data collected from Esra and Bilge (pseudonyms) who created a common lesson plan. We selected Esra and Bilge as participants because they showed different characteristics in terms of their lesson plan, eagerness and critical view to teaching, comparing other prospective teachers. The prospective teachers were asked to answer the four open-ended questions, which are indicated in Table 2. Each individual interview lasted for approximately 15 minutes. When necessary, prompts were employed during the interview in order to enable the prospective teachers to provide concrete examples. The interviews were recorded by a video camera. The data collected from the interviews were coded by using descriptive analysis.

1. How necessary and important is being mathematically creativity for prospective mathematics teachers?

2. What kind of activities should a teacher carry out in order to be mathematically creative for you?

3. What should a student do in the classroom in order for you to consider his/her actions as a mathematically creative behavior/action?

4. What sorts of activities are appropriate to activate the mathematical creativity of the students by carrying out activities in the classroom?

Table 1: Interview questions

FINDINGS

Characteristic and Practices of a Creative Teacher

The prospective teachers defined the characteristics of a creative teacher as ''attention-grabbing, being able to use his/her tone of voice effectively and being attentive while approaching the students''. They mostly described the creativity in education through the activities that the teachers implemented. These activities may include using various resources, making lessons interesting for the students, making group projects in order to increase the interaction among the students, creating B plans for successful students, assigning research projects to students (Table 2).

Using various resources (e.g. real life problems, open-ended problems, manipulatives)
Making lessons interesting for students (e.g. drama, puzzle)
Making group projects in order to increase the interaction among the students
Creating B plans for successful students
Assigning research projects to students
Characteristics of a teacher (e.g. attention-grabbing, effective tone of voice,

Table 2: Characteristic and Practices of a Creative Teacher

Esra proposed to create and use B plans for the students who are a step further and / or more eager than their friends in the classroom activities. Moreover, she considers creativity as a means to be used while teaching, not as a mathematical content (Kattou, Kontoyianni, & Christou, 2009). Her explanation on this is as follows:

Even if I wanted to explain a notion creatively, I mean a triangle or a point has a certain description. I mean I have to explain it in that certain way. But the way I present it, using inverted sentences, explaining it in another way, or using materials or explaining by using drama, these are the creative tools that I use.

Esra stated that she could teach the lesson through using drama by activating her creativity. In another explanation, she mentioned that she could use puzzles in order to ''grab attention to the lesson''. Similar to Esra, Bilge evaluated creativity within the context of education activities. Bilge considers an activity, prepared by using various resources such as real life problems and manipulatives, to be a creative activity. She explains this by the following statements:

> I use different books, create my own problems, and associate them with real life, that is how I use these problems. Manipulative, for example, I used geometry board yes but I will not need it for every problem, maybe for some.

Bilge described creative activity as stated above and discussed that activating the interaction among the students by group working is a creative activity. She also included assigning research projects to the students outside the classroom besides in-class activities, within the scope of creative education.

When the prospective teachers were asked to give examples for mathematical activities, which activate the mathematical creativity of the students through in-class activities, they mostly gave explorative or investigative activities as examples. For instance, Esra said the following regarding a problem, which she found interesting: ''One of our teachers drew a *star* in the class and asked if it was a polygon. It was an outside-the-box demonstration for us, all of these can be mathematical creativity.'' Furthermore, another creative activity example of Esra is the manipulative example which she and her friends used in the lesson plan they had created and it is indicated in the Figure 1.

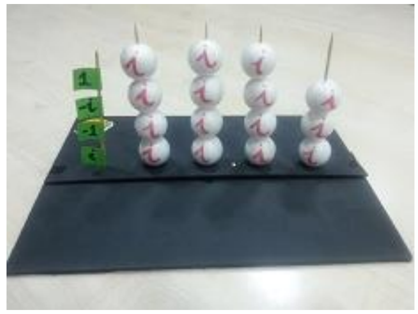

Figure 1: Manipulative example that Esra gave as a creative activity

Characteristics of a Creative Student

The prospective teachers described the characteristics of a creative student as associating the subject with other subjects, proposing different solutions and approaches, asking challenging questions (Table 3).

Associating the subject with other subjects
Proposing different solutions and approaches
Asking challenging questions

Table 3: Characteristics of a Creative Student

Esra described a student who proposes different solutions and solves the problem by associating with different subjects as mathematically creative and her description is as follows:

> For instance, there are some students who propose different solutions to the problems in order to see if they can solve it by using those. In addition, they found right or wrong solutions by themselves and they associate it with other subjects and focus on certain points while explaining it. For example, while I teach them as y=x line, one of the kids says bisector line, the first bisector line. These are the things they know but he might have activated this mathematical creativity by solving it by associating it with other subjects.

Esra considers the student's statement about y=x line being the first bisector line at the same time during the ''school experience'' and thus associating with different subjects, to be an indicator of mathematical creativity. On the other hand, Bilge stated that she could describe a student who investigates as a creative-thinking student:

> The students will be satisfied and will not have any questions in their head and will meaningfully learn the subject if the teacher briefly explains, of course without getting into detail, the questions the students ask such as where did it come, why it is formed in this way, why do we need to learn derivatives or why is it like that, the questions they ask about the nature of mathematics and the knowledge itself. Or students' asking questions within this context, mathematical context could be creative thinking, in my opinion.

As seen in her statement above, Bilge considers investigating the necessity of mathematical objects as an indicator of creativity. Besides, it is observed that Bilge describes creative students as a student who has flexibility features. Her explanation on this is as follows:

> Solving the problems by using different approaches, different ways, I mean it is really nice, there are at least 30 students in the class and it is really nice that there are at least 2 or 3 different solutions, of course that depends on the problem or the subject. It would be nice to give the student a chance by calling him/her to the board saying that 'your friend solved the problem in a different way, let's see his/her solution'.

Considering the descriptions of Esra and Bilge on the characteristics of creative students, it is observed that they explained it through problem solving. They explained the characteristics of creative students based on the approaches of the students to problem solving. Moreover, based on the collected data it is possible to say that the students consider creativity both as a process and a product.

DISCUSSION

According to findings on the views of Esra and Bilge about the characteristics of a creative student and teacher , it is possible to say that dominantly what prospective teachers think about mathematical creativity is not the mathematical content itself, but rather the teacher and his/her practices (Kattou, et. al., 2009). Creativity in school mathematics obviously varies from the creativity of mathematicians (Sriraman, 2005). However, it was determined that the prospective teachers' views on creativity are limited to the activities prepared by the teachers and the students' approaches to problem solving. It is concluded that the most effective determinant for this is due to cultural and contextual factors. Nowadays, in Turkey the teachers mostly give direct instructions. It is observed that the prospective teachers considered the activities and tasks a teacher, who does not give direct instructions, could use, as creative. They also focused on the importance of different activities that could be used in classes while explaining the characteristics of a creative teacher (Sheffield, 2006; Mann, 2006). Consequently, it is determined that the prospective teachers do not have detailed information on creativity. The reason for this is due to the fact that they did not have a lesson about creativity throughout their teacher education. What the students primarily need in order to activate their creativity is having pedagogical knowledge related to creativity (Shriki, 2008). Hence, it is assumed that it will be made possible through the lessons they will have throughout their teacher educations programs. Teacher education programs should be designed in a way that they support prospective teachers to acquire knowledge and teaching skills on the development of students' creativity (Hosseini & Watt, 2010; Levenson, 2013). With the findings of this study and the future studies, it is planned to create a lesson about creativity in the teacher education program in Turkey.

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ABILITY GROUPING FOR MATHEMATICALLY PROMISING STUDENTS

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Abstract. *Grouping mathematically promising students is one of the foundations of exemplary gifted education practice. The research and best practices show a positive outcome of this type of practice. This paper will be revealing myths around ability grouping and presents empirical findings that full-time ability grouping generates significant academic improvement and encourages creativity. Policy makers need to adapt to ability grouping practice in curriculum development in order for the highest achieving students to maximize their mathematical talent.*

Key words: Mathematically promising students, Ability grouping, Policy for gifted students

BACKGROUND

The U.S. school system is not fulfilling the needs of the most capable students, in particular mathematically promising students. The ideal mathematics curriculum must meet the needs of individuals of all levels. According to NCTM's Statement and Belief (1998), "Every student deserves an excellent program of instruction in mathematics that challenges each student to achieve at the high level required for productive citizenship and employment" (p. 5). *An Agenda for Action* document reported that the U.S school systems failed to identify mathematically promising students and did not implement the appropriate programs for those students. The report also stated that "The student most neglected, in terms of realizing full potential, is the gifted student of mathematics. Outstanding mathematical ability is a precious societal resource, sorely needed to maintain leadership in a technological world "(p.18). In order to improve our school systems and student achievement, two main goals must be met: equality and excellence. Equality refers to the opportunity of every learner to have access to the highest possible education. Excellence refers to every learner's need for opportunities and support to maximize their learning potential (National Middle School Association, 2005).

In this paper, the following position will be discussed: In order to achieve equality and excellence for mathematically promising students, ability grouping should be included in a common curriculum practice in the U.S. school system. Based on several studies and research, the use of both tracking and ability grouping in the U.S. school system has been interchangeable. The common theme in both terms is the separation of students based on their performance and their apparent ability. In addition to these expressions, we will be using the terms *homogenous* and *heterogeneous* grouping practices. For a long time, researchers have tried to answer questions about whether mathematics students are challenged at appropriate levels according to their needs in different types of educational settings (e.g., homogenous, heterogeneous). Some researchers and educators *assume* that non-tracking or non-ability grouping is the best practice for school children. In the next paragraphs, the three main issues will be addressed.

HOMOGENEOUS VS. HETEROGENEOUS

According to Usiskin (1999), the most widely cited publication against the tracking practice is *Keeping Track: How Schools Structure Inequality* by Jeannine Oakes (1985). She completed her research on tracking in 25 representative schools. Oakes (1985) claims that the learning of average and slow students can be negatively affected through the same ability grouping practice. In her strong opinion, "no group of students has been found to benefit consistently from being in a homogenous group" (Oakes, 1985, p. 7).

NEGATIVE CONSEQUENCES

Another major criticism of ability grouping is that it would lower the self-esteem of students in low-ability groups. The supporters of this idea claim that students placed in average and low-track classes do not develop positive attitudes. Students in the upper-track, on the other hand, sometimes develop "inflated self-concepts" as a result of their grouping or tracking (Oakes, 1985).

SOCIAL INEQUALITY

The third and most controversial issue in homogenous ability grouping is equality and discrimination in the school system. Poor and minority students regularly score lower than white students, and based on these scores, they are placed in the bottom groups, whereas white students are more often at the top. In 1995, Oakes's study conducted in two urban city districts found that both schools incorporated "racially imbalanced classes" in all grade levels. In another study on mixed-ability grouping, Boaler (2008) supports a heterogeneous or mixed-ability classroom setting and an associated set of teaching practices. In her study, she found that heterogeneous grouping practice allowed students to interact with others from different social classes, cultural groups, and ability levels. In terms of social interaction, students respected and valued the diverse ways in which they approached their mathematics. Finally, supporters of the anti-ability grouping movement use Hanushek's (2006) study that compared student's early grouping or tracing and international performance tests results. The study results suggested that early tracking increases educational inequality.

ACADEMIC BENEFITS AND ABILITY GROUPING

Several studies indicated that ability grouping had been shown to raise academic standards (Tieso, 2003; Rogers, 2002; Clifford, 1990; Anderson & Pellicer1990; Kulik, 1993). Ability grouping allows a focused curriculum and appropriately paced instruction that leads to the maximum learning by all students. Overwhelmingly, evidence supports policy that recognizes the need for flexible and homogeneous grouping for targeted instruction with a curriculum matched to a student's aptitude in mathematics (Stambaugh & Benbow, 2010). One type of ability grouping is a subject acceleration. Kulik's (1992) study showed that highly talented youngsters profit greatly from work in accelerated classes. The researcher stated, "Schools should therefore try to maintain programs of accelerated work" (Kulik, 1992, p. 25). U.S. school system's goal should be to expand, rather than restrict, the academic and social environments of the promising students. Ability grouping can provide a planned pathway to a developmentally appropriate placement.

Kulik (1993) reviewed two sets of meta-analyses on research findings on ability grouping. All of the studies in both meta-analyses confirmed that higher ability students usually benefitted from ability grouping. Grouping was found to have less influence on the academic achievement of the middle and lower ability students. The researcher compared these findings to the well-known Oakes (1985) study. According to Oakes, students in top tracks gain nothing from grouping and other students suffer obvious disadvantages, including loss of academic ground and self-esteem. The review concluded that American education would be seriously harmed by the wholesale elimination of grouping programs.

Grouping is a vehicle that educators can use to allow mathematically promising students access to learning at the levels and challenge they need. Tieso's (2003) study found that flexible ability grouping, combined with appropriate curricular revision or differentiation, may result in substantial achievement gains both for average and high ability learners. Tieso (2003) found that different curriculum with high-ability students resulted in significant achievement gains compared with the regular mathematics curriculum. This finding was supported by Roger's (2002) study. She synthesized 13 research studies on ability grouping and concluded that high-ability learners need some ability grouping in order to effectively and efficiently accomplish their educational goals. Rogers' (2002) findings show substantial academic effects (anywhere from 1 1/3 to 2 years' growth per year) when gifted children are grouped together full-time (Rogers, 2002). She also addressed some of the other legitimate criticisms saying that "one size does not fit all, whether that solution involves mixed-ability classroom conformations or ability groping in one or many of its forms" (p. 107).

National data shows great outcomes, with an increased number of high school students successfully passing AP exams in calculus. Without ability grouping and homogenous mathematics classroom practices this positive result would not have happened. Johnsen and Sheffield (2012) expressed, "As these numbers have increased, more students now receive credits for first–year calculus in high school than in college" (p. 10). Furthermore, looking at the progress of talented U.S. mathematics students over time, and comparing them with their international peers, shows evidence of determination of the brightest students. Gieger and Kilpatrick (1999) analyzed U.S. students' achievement levels in the International Mathematics Olympiad (IMO). They found that the results were very promising and the U.S mathematics team always placed in the top five (p. 33). Even though the sample was too small to draw any conclusions, they added that, "these results indicated that at least some of our most promising mathematics students do excel in an international setting" (p. 33). Similarly, U.S students who were studying AP calculus did score above the international average among their peers (p. 35).

SOCIAL BENEFITS AND ABILITY GROUPING

Ability grouping provides some students with the opportunities to make their self-esteem higher. More importantly, it permits children to learn and make social connections with students who think and learn in the same ways they do. When we think of students' emotion and motivation, Gross's (2006) study found that several of the non-accelerants cannot recall their school years and accepted it as a painful part of living. By contrast, young people in this study who were accelerated by 2 or more years believe that they were now more appropriately placed in terms of their academic, social, and emotional needs.

Similarly, Benbow, Lubinski, Shea, and Eftekhari-Sanjani (2000) examined 1,975 mathematically gifted adolescents, ages 13 through 33. 33-year-old accelerated participants were asked some questions about how their view of ability grouping has affected their educational, professional, and social planning. They clearly indicated that educational acceleration had its most helpful effect on their educational and professional planning. Acceleration benefited them by increasing their interest in education, enhancing their involvement with extracurricular activities, and preparing them to begin contributing to society at an earlier age. One part of the study measured groups' attitude toward ability grouping. Both sexes' attitude toward eliminating homogeneous ability grouping for instruction was the same. Almost 80% of them were unsupportive of the proposal. Participants were very much against eliminating homogeneous grouping for instruction. Clifford (1990) reviewed some slow-track programs and concluded that students who enter special education programs are likely to remain in the same level for many years, often for their entire academic careers. To overcome this problem, the researcher gave the following suggestions: raising expectations, endorsing higher order thinking skills, and using more active teaching methods.

ABILITY GROUPING AND ENHANCED EQUALITY

The word equality means the opportunity for all students to develop their mathematics abilities to their full potential. All students should have opportunities to grow and develop, and equality means that this fundamental idea also applies to students who are gifted in mathematics (Warshauer, McCabe, Sorto, Strickland, Warshauer, & White, 2010). Basically, every student should have the opportunity to learn something new every school day. Without ability grouping, it is hard for educators to live up to these expectations.

Opponents of ability grouping usually refer to the Hanushek (2006) study. The results of it suggest that early tracking increases educational inequality. There is actually a huge flaw in this study. For one, it compared two different sets of data. The first was the 2001 PIRSL test for 4th graders, ages 9 and 10, and the second was the 2003 PISA test for 15-year-old students. Therefore, the participants were not in the same populations. Thus, we can conclude that the study result is statistically unreliable

CLOSING REMARK

When a new policy document or urgent report on mathematics education is released to the public (e.g., NCTM, 2000; CCSSM, 2010), the underlying reason for these urgencies for change is mostly the U.S students' poor international performance scores. There is evidence that shows a strong positive correlation between students' early grouping and international performance tests. Plucker, Hardestyand, and Burroughs's newly released report (2013) supported previous studies' results. They concluded that "the percentage of American students who performed at advanced levels on TIMSS and PIRLS does not compare favorably to those in other developed countries" (p. 22).

Similarly, within the U.S., the researchers found little evidence of shrinking gaps. In most cases, gaps have stabilized or grown, and levels of advanced achievement and the size of excellence gaps vary considerably across states. In terms of policy making, ability grouping should be included in a common curriculum practice in the U.S. school system for mathematically promising and talented students. Policies should rely on research evidence that supports the best practices for this population. Overwhelmingly, evidence supports policies that recognize the need for flexible, homogeneous grouping for targeted instruction with a curriculum matched to a student's aptitude in mathematics. When any new education policies are created, policymakers should ask themselves questions about the new policy's impact on the top achieving students and work toward helping more students achieve at the highest levels.

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SUPERMATH: A CREATIVE WAY TO ENGAGE TALENTED MATH STUDENTS

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Abstract. *This article describes a new pedagogical method for engaging mathematics students in creative thinking. Students in an eighth grade advanced mathematics class were assigned the task of creative a math-focused comic book. Students were divided into groups and asked to create a superhero who had math-based superpowers. The groups then wrote a story in which the superheroes had to work together to solve a problem that involved an advanced mathematical concept. The students wrote and illustrated the comics collaboratively. The article begins with an examination of comic books as a teaching tool. It then describes the activity assigned. Samples of the student work are then presented along with a discussion of the procedure used for assessment of the math comic books. The article concludes with a summary discussion of findings.*

Key words: comic books, superheroes, collaboration, mathematically talented students

USING COMICS TO TEACH MATHEMATICS

In recent years, the use of comic books has become more common in classrooms. While some educators are resistant to this popular cultural medium, there is also growing support for the use of comic books as a teaching tool. In this article, I will begin with a brief examination of the use of comic books for teaching. I will then describe an assignment that asked students to create math focused comic books. I will discuss several samples and describe the assessment method used for the activity.

Gone are the days when students got in trouble for having a comic book tucked away between the thick pages of a textbook. Popular culture is a key component to students' lives today. As a result, teachers should be willing to consider comic books and graphic novels as a tool that can be used to motivate students to be actively engaged in the learning process. Hall (2011) states that "comic books and graphic novels are one of the newest fully fledged art forms, a vibrant hybrid medium birthed in America and brimming with all the wildly experimental vigor of youth" (p. 39). The use of comic books can capture a student's imagination and allow teachers to combine content with context while at the same time getting students to reflect on both math concepts and social issues in the comic books.

Comic books can be used to engage and motivate students who are gifted and talented in mathematics. Matsko and Thomas (2014) point out that "motivation in mathematics is of particular concern for educators in both gifted and general populations" (p. 155). In research carried out by Martin and Pickett (2013), gifted math students reported that their work was generally too easy despite their teachers reporting in the same study that the work was challenging.

It is also important for teachers to consider how math learning can be placed in meaningful, appropriate, and creative settings. Comic books provide an excellent opportunity for bringing creativity and novelty into the gifted mathematics classroom. Cooper, Nesmith, and Schwartz (2011) interviewed educators from the elementary to university level who used comics as a teaching tool. These researchers found that comics could "promote higher order thinking, offer realistic connections and examples, and some real world activities" (p. 6).

Such an approach can be particularly beneficial in classes when working with students who are mathematically talented as it allows the students an opportunity to go above and beyond the activities in a typical math class. By having students use and create comic books in math class, students can bring their own ideas and experiences into a mathematics lesson, thereby using their personal backgrounds and abilities to aid in the development of their mathematics understanding. Silver (1997) discussed the idea of having tasks and activities that "increase students' capacities with respect to the core dimensions of creativity, namely, fluency, flexibility, and novelty" (p. 75). By providing gifted and mathematically talented students with the opportunity to try novel activities, such as math comic book writing, teachers can promote higher order thinking.

CREATING MATH COMIC BOOKS

A math assignment was designed which asked students to write a comic book in which the characters had math-based superpowers. Students were assigned to collaborative groups and each member of the group had to create a character who had some type if a mathematical talent. The group was then asked to think about a problem these characters would have to solve using their mathematical powers. These stories were to be written in comic book format including illustrations and dialogue balloons.

The comic book writing assignment was used in an eighth grade advanced mathematics class. There were 10 students in the class. These students had been together since sixth grade and so they had a very good working relationship. The idea of working collaboratively was not new to them. In fact, because of their advance mathematical skills, they were excited to try something different. By writing comic books about mathematical concepts, the students were challenged to find a way to frame their understandings of the concepts. That is, they would have to tell and illustrate a story which showed they understood a specific mathematical idea. The students would also have to link that idea to the story's plot, thereby demonstrating how mathematical knowledge can be related to real life situations.

The class was divided into three groups and created three comic books with various themes. All of the comic book adventure themes involved heroes working together to prevent some disaster from occurring. In *Math Men*, the superheroes had to rescue a child from a dungeon by helping the child understand multiplication. In *The Subtractors*, the superheroes had to disarm a bomb by breaking a code. And in *The Adventures of Supercow and Frost*, the heroes had to find the area of an irregular polygon to protect their city from a powerful sunbeam.

MATH MEN

The authors of this comic book were three male students. The characteristics of the superheroes in this story are clearly linked to the traits of middle school age boys. In all three cases, the superheroes parents' were dead and they had been taken in by a kindly stranger who helped them develop their ability to use their powers. When asked about why the parents were deceased in the story, the boys talked about minor battles they had been having with their parents. These were battles like not wanting a curfew and being able to participate in a sport their parents thought would be dangerous. The descriptions of the characters' lives before they met the stranger were that of being alone and not really fitting in. This was also true of the authors' real lives. Because these students were talented in math and had been in accelerated math classes since elementary school, they were often seen as part of the nerdy group and so did not hang around with the popular students. While they brushed this issue off in real life, the comic book story gave them an opportunity to elaborate on this bias, to find others in the same situation, and to create a group of friends that got along because they were different rather than being shunned for being different.

The adventure in Math Men was to have the superheroes rescue the 9-year-old niece of the stranger who had helped them with their powers. They selected a math topic that would be familiar to a 9-year-old, since she was the one looking for help. The students selected multiplication as their math concept. While multiplication was obviously not as challenging as the math these students did in their regular math class, they did put a twist on it by looking at different strategies for doing multiplication (partial products and lattice) rather than just using the traditional multiplication method. Because these students had been accelerated throughout their elementary years, they would have been taught multiplication using the traditional method. However, the students all had younger brothers or sisters who had recently been introduced to these new methods for doing multiplication, and so these boys decided they would like to tie that into their story by having to help a third grader understand multiplication. Once the girl was able to solve the multiplication problems, she would be free to go. (Appendix 1.)

THE SUBTRACTORS

The authors of *The Subtractors* were two male students and two female students. When describing their backgrounds before becoming a team, these superheroes still talked about not being part of a group, but they also included what they could do. For example, one superhero had amazing tennis and dance ability, another had been part of a great soccer team, and another had the ability to work with animals. These characters were not separated from their families but rather were called upon from separate activities after their amazing talents were spotted. Because this group was more diverse that the previous group and had one more student, there was more activity throughout the story as can be seen in the math piece selected.

The superheroes in *The Subtractors* had to find and disarm a bomb by breaking a code. While these students may have done math activities in previous years on finding codes, it was not part of their then current math class. Therefore, these students had to research what kind of a code they wanted to use in their story. After much discussion during work time, they decided to go with a Vigenere Cipher which is more challenging than a Caesar Cipher. The students said they thought the Caesar Cipher seemed a little too straight forward. Unlike *Math Men*, this story did not involve a third party without a superpower, and so the students used the opportunity to enhance their own mathematical skills by researching different types of ciphers and deciding to use this comic book activity to examine a topic that was normally not covered in their regular math class. Because of the additional member in this group, there were a lot more speech balloons throughout the comic book and there always seemed to be two people talking at once. While some might find that a little distracting at times, this does show that these students were starting to see the value of others and the benefits of having a diverse group even if the group members' mathematical interests were similar. (Appendix 2.)

THE ADVENTURES OF SUPERCOW AND FROST

The authors of the third comic book were three female students. Unlike the other two groups, this group did not provide any background about their characters. The three female superheroes were from the same neighborhood and while two were friends prior to meeting for this adventure, there was no discussion of previous meetings or encounters. One character had a watch that could detect crime and on the way to the crime scene or what they thought was a crime scene, they met a third superhero, Monkeyous. In *The Adventures of Supercow and Frost*, an evil villain was about to use a mirror to direct a powerful sunbeam to destroy part of the city in which the three superheroes lived. The superheroes needed to build a shield that would protect their city from the sunbeam. The story in The *Adventures of Supercow and Frost* was not as linear as that of the previous two comic book stories. At any given time, one or all of the three superheroes would for no reason suddenly change into their animal characters as they worked to come up with a solution. In another odd twist to the story, the superheroes found and had to prove the candy was just M&Ms and Skittles and not memory-erasing pills. Although entertaining, the writing in this story could certainly have used further revision.

This group of students had selected a math topic which most closely related to what the students were studying in their current math class. This topic dealt with the area of polygons. However, the students decided to extend this math beyond just using the formula for finding the area of a regular polygon. In order to make the story more realistic, the superheroes in the story needed to build an irregular shaped shield rather than a regular shaped shield. While finding the area of irregular polygons had been discussed in their math class, the students researched a different way of finding the area by using coordinates and adding or subtracting the areas of different trapezoids created by their sides and the x-axis. Like the student authors in *The Subtractors*, these students used this activity as an opportunity to extend their study of a math topic that would not have been introduced in as much detail in their regular class. (Appendix 3.)

EVALUATION AND FINDINGS

A rubric (Figure 1) was created to evaluate the students' work. While the rubric was handed to the students at the beginning of the project, they were given feedback throughout the project on ways to improve or clarify their work. Students were asked to submit drafts of their character traits and the storyplot at various points. This gave the students opportunities to make changes before they started to write the comic book and create illustrations.

Rubric for Comic Book Assignment			
Completeness	Elaborate story created and complete matching illustrations	Story created but does not elaborate on details. Some matching illustrations	Story is not complete or difficult to follow. Illustrations incomplete and not matching
Writing Skill	Story and illustrations well organized and situations clearly explained	Story and illustration lack clarity in places	Story missing important information or elements to convey to situation
Creativity	Ideas shared are original, humorous, and sophisticated	Most of the ideas are original and humorous	Ideas are not original
Visual Appeal	Multiple backgrounds, characters, and props are used	Some backgrounds, characters, and props are used	Limit use of backgrounds, characters, and props
Mathematical Complexity	Challenging mathematics concepts fully explained and used in appropriate ways	Mathematics concepts introduced but not always used appropriately	Mathematical concepts not clearly used

Figure 1: Rubric used to evaluate the comic book activity.

The gender breakdown of the superheroes in each group was the same as that of the authors. When asked to create characters all groups made their characters about 10 years older than themselves. Nevertheless, the groups gave their characters the same characteristics and traits as their 13-year-old selves. The all-male group tended to have a darker story. That group's characters backgrounds seemed to be sadder than the backgrounds of the group that had a combination of male and female students. There was also a lot more fighting in the all-male authored story than in the other two comic book stories. In the two groups that had female students, animals played a role although in both cases it was not a significant role.

There was also a lot more dialogue among the characters in comic books written by groups of female students. This talk was often centered on getting various ideas about how to proceed to solve the problem and trying to pick the best solution. The comic book activity gave these often quite female students an opportunity to talk together. The female characters in the comic book story that had both male and female superheroes spoke much more freely and frequently than their counterparts in the classroom. The comic book activity appeared to give these mathematically talented female students a voice and confidence to discuss mathematical concepts in front of their male peers.

CONCLUSION

The project described in this article set out to provide mathematically talented students with an opportunity to work with mathematical concepts in creative ways. While the students were brainstorming their characters and stories, the student groups had lively discussions about different ways mathematics can be used to solve problems. The students seized this opportunity to examine mathematical concepts outside of their regular curriculum. This research matches work done by Matsko and Thomas (2014) who reported that talented mathematics students are intrinsically motivated when they feel they have a sense of control and ownership over their own learning. In this activity, the idea of presenting the work in a comic book format was given to the students, but it was the students who had to come up with the characters, the characters mathematical talents, and the math problem that the characters had to solve.

This project encouraged students to work with challenging mathematical tasks. Students researched math topics that went beyond those they had studied in their regular math classes. Topics were not suggested to the students. Instead students independently researched math concepts and then worked to incorporate these concepts into their comic book stories. By encouraging further exploration of math topics, this project actively engaged students in new kinds of learning. Even students that were normally quite passive during their regular math class became excited by working on this project. The project engaged these mathematically talented students by giving them the kind of autonomy gifted learners desire. This comic book writing project became a super way to teach students about math.

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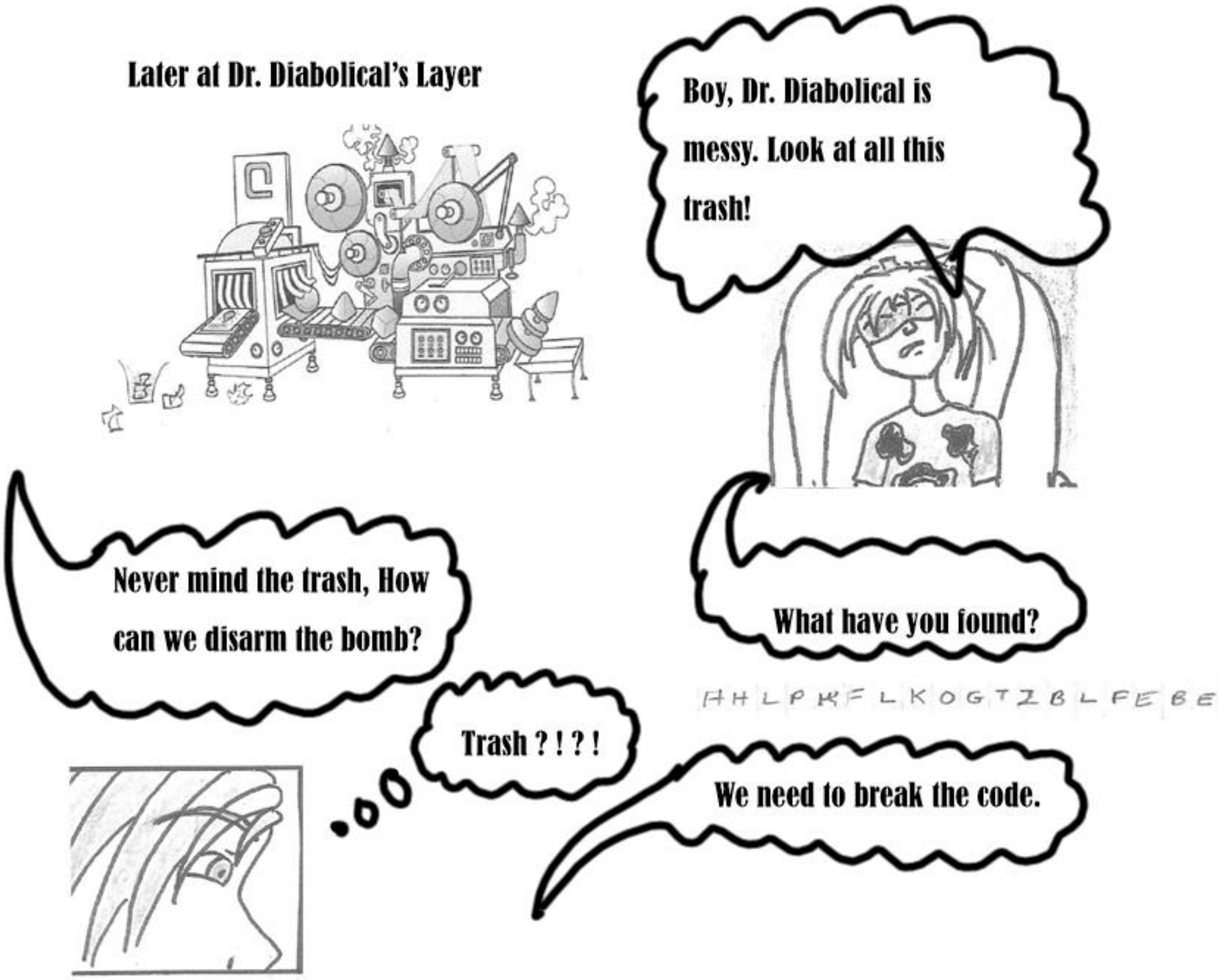
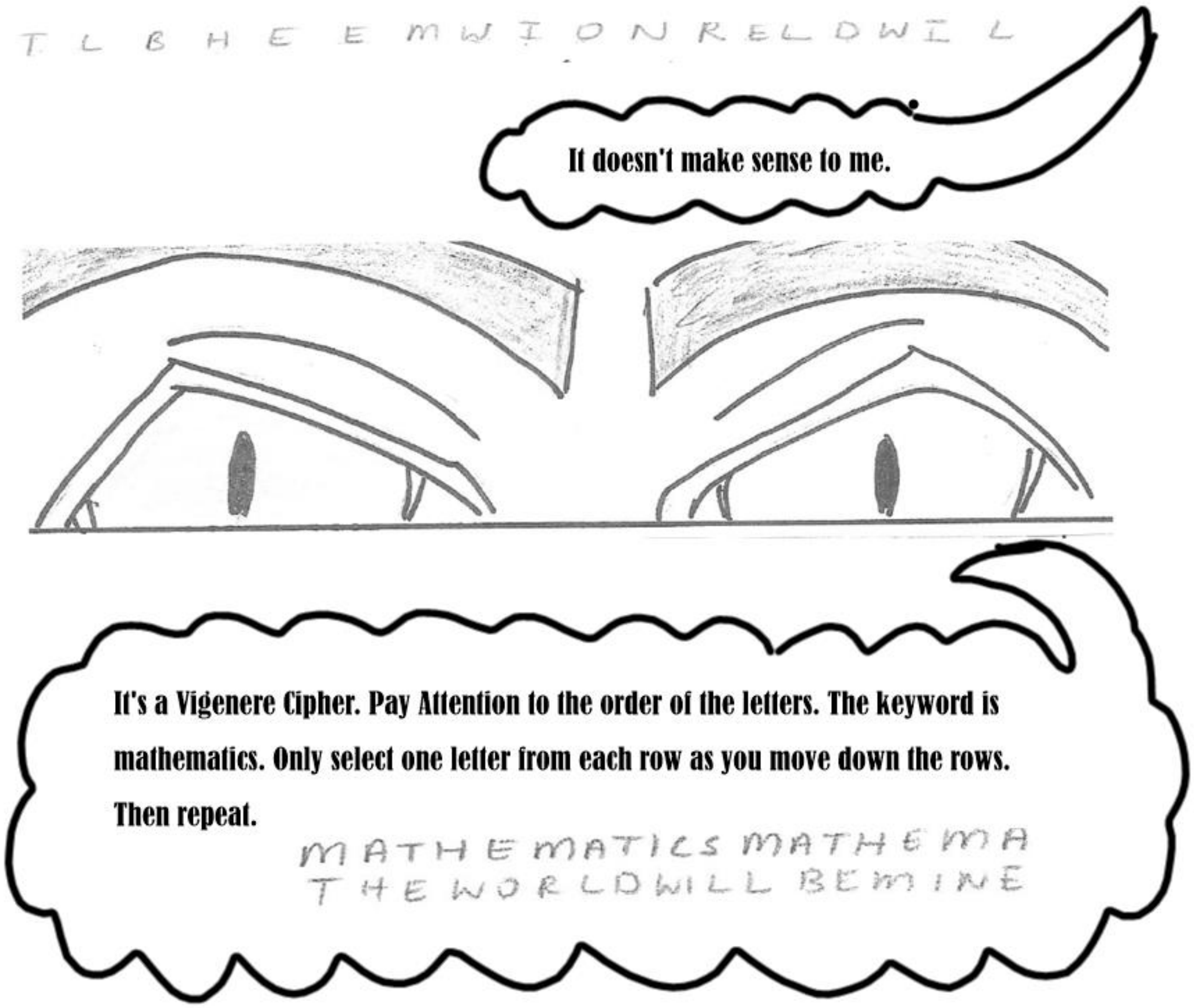
Appendix 1—Math Men

Appendix 2—The Subtractors

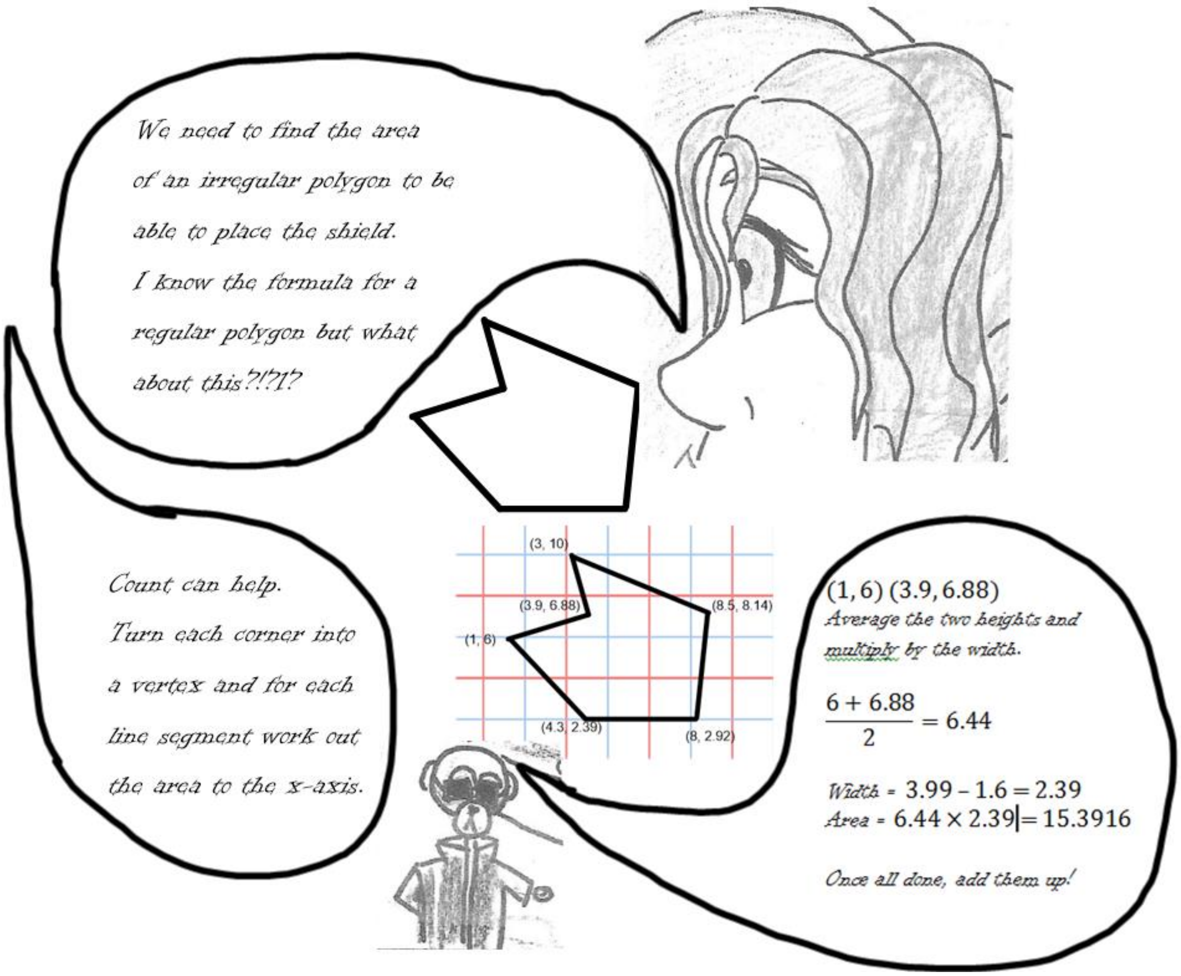


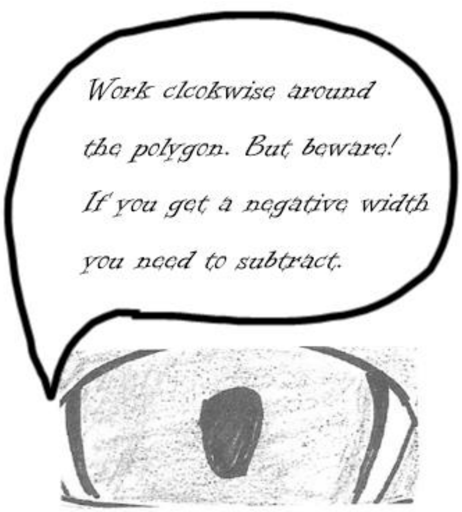
A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L
A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S
H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G
E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D
M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L
A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S
I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H
C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z	A	B
S	T	U	V	W	X	Y	Z	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R

Appendix 2—continued


T L B H E E M W I O N R E L D W I L
It doesn't make sense to me.
It's a Vigenere Cipher. Pay Attention to the order of the letters. The keyword is mathematics. Only select one letter from each row as you move down the rows. Then repeat.
MATHEMATICS MATHEMA
THEWORLDWILL BEMINE

Appendix 3—The Adventures of Supercow and Frost





From		To		Ave height	Width =/-	Area +/1
x	y	x	y			
1	6	3.9	6.88	6.44	2.39	15.3916
3.9	6.88	3	10	8.44	-0.9	-7.596
3	10	8.5	8.14	9.07	5.5	49.885
8.5	8.14	8	2.94	5.54	-0.5	-2.77
8	2.94	4.3	2.39	2.665	-3.7	-9.8605
4.3	2.39	1	6	4.195	-3.3	-13.8435
					Total	31.2066

ORAL PRESENTATIONS 3.3.

Teaching discourse in creative learning environments

Chair of the session: **Ralf Benoelken**

DO TEACHER'S WAYS OF ENHANCING DISCOURSE IN HER CLASS LEAVE TRACES ON HER STUDENTS' POST-TEST RESPONSES?

Rina Hershkowitz	Michal Tabach	Shirly Azmon	Chris Rasmussen	Tommy Dreyfus
The Weizmann Inst. for Science	Tel Aviv University	Levinsky College of Education	San Diego State University	Tel Aviv University

Abstract. *The goals of the current study are to investigate if and how the genre of the mathematics teaching-learning discourse in the whole class setting has some longitudinal effect on individual students' knowledge as expressed in a post-test. Specifically, we observed elements of creativity in the whole class discourse and their echoes in students' responses. With this aim we analysed probability learning in two eighth grade classes. We present here data analysis from one lesson and one corresponding post-test item from each class. The differences found in the analysis of students' responses to the post-test item reflect differences in the genre of the whole class discussions brought about by the two teachers. While one teacher opened opportunities for students to express their creative explanations, the other one did not.*

Key words: whole class discussion; teachers` creative prompts; eliciting creative ideas from students; individual responses echoes classroom discussion

INTRODUCTION

During the last few years we have been investigating knowledge construction and knowledge shifts among different settings in the classroom: the individual, the small group and the whole class community. In these investigations, we used two theoretical-methodological frameworks – Documenting Collective Activity (DCA) for investigating the whole class setting and Abstraction in Context (AiC) for investigating individuals and small groups (Hershkowitz, Tabach, Rasmussen & Dreyfus, 2014; Tabach, Hershkowitz, Rasmussen & Dreyfus, 2014). Both frameworks focus on construction of mathematical knowledge in an inquiry based class. The DCA analysis helps illuminate what is happening on the social or discursive plane, while the RBC+C analysis helps illuminate what is happening on the cognitive side. As such, the process of constructing new knowledge seems to include creative processes, where creativity is taken in its most frequent (popular) meaning: the ability to transcend traditional ideas, rules, patterns, relationships, and the like, and to create meaningful new ideas, forms, methods, interpretations, etc.

The goals of the current study are to investigate if and how differences in findings between identical post-test's task of two classes can be explained by the genre of the teaching-learning discourse in the whole class setting? Meaning that, these discourses had some longitudinal effect on individual students' knowledge as expressed in a post-test, and to try analyzing and interpreting the findings through the lens of creativity.

THEORETICAL BACKGROUND

Argumentation and learning from a socio-cultural perspective

A socio-cultural perspective helps us appreciate the reciprocal relationship between individual thinking and the collective intellectual activities of groups (Vygotsky, 1978).

Argumentative talk is the main vehicle by which we transform individual thought into collective thought and action, and conversely to make personal interpretations of shared experience. Generally, creative thought and argumentative talk have a crucial role for school learning (Hershkowitz & Schwarz, 1999).

Research shows that quite often, argumentative talk is not part of classroom mathematical talk or focus on students' and teachers' creativity. The dominant genre of talk consists of recitation style discourse patterns such as Initiation-Response-Evaluate (IRE) (Cazden, 2001). Moreover, teacher interventions in teacher-led classroom discourse are often not tied to students' ideas. As Yackel (2002) showed, to tie her interventions to the students' ideas the teacher must first identify the students' threads of thought, and then find a way to advance their reasoning. Some researchers have proposed that teachers provide *generic prompts* (e.g., prompts for encouraging argumentation and creativity, mostly prompts expressed as questions, Mercer, 2000), that creatively break the IRE patterns and bring the classroom talk closer to argumentative forms of talk and enhanced the creativity potential.

Abstraction in Context and Documenting Collective Activity

Abstraction in Context (AiC) is a theoretical framework for investigating processes of constructing and consolidating a new mathematical knowledge (Hershkowitz, Schwarz, & Dreyfus, 2001). In AiC, abstraction is defined as an activity of vertically reorganizing (Treffers & Goffree, 1985) previous mathematical constructs within mathematics and by mathematical means, interweaving them into a single process of mathematical thinking so as to lead to a construct that is new to the learner.

Collective Activity is a sociological construct that addresses the constitution of ideas through patterns of interaction and is defined as the normative ways of reasoning which have developed in a classroom community. Such normative ways of reasoning emerge as learners solve problems, explain their thinking, represent their ideas, etc. A mathematical idea or a way of reasoning becomes normative when there is empirical evidence that it functions in the classroom *as if it were shared*. The phrase "function as if shared" is similar to "taken as shared" (Cobb & Bauersfeld, 1995) but is intended to make a stronger connection to the empirical approach which uses Toulmin's (1958) model of argumentation to determine when ideas function in the classroom as if they are mathematical truths (Rasmussen & Stephan, 2008).

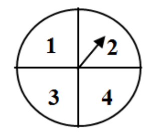
The concepts of knowledge agent, and uploading and downloading of ideas

A *knowledge agent* is a member in the classroom community who initiates an idea, which subsequently is appropriated by another member of the classroom community (Hershkowitz, et al., 2014; Tabach, et al., 2014). Thus, when a student in the classroom is the first one to express an idea according to the researchers' observations, and others later express or use this idea, then the first student is considered to be a knowledge agent. We may here ask several questions: Is the student who is a knowledge agent the most creative one in the given episode? Does the fact that there are other students who follow this student evidence that this student has creative ideas? As Leikin & Pitta-Pantazi wrote: "Creative ideas are those that are considered by the reference social group as new and meaningful in a particular field" (p. 161).

In the present study we first focus and analyze differences in findings of individual students' post-test responses from two classes. Next we analyze the collective discussions of the whole classes seeking its traces in the individual students' knowledge, as it is expressed in post-test responses. We ask: Do the differences in students' responses in a post-test between two classes reflect, at least partially, differences between whole class discussions led by the teacher? Are these differences related to the creativity of the teacher in leading the discussion?

METHODOLOGY

A 10-lesson learning unit in probability was implemented and video-recorded in several eighth grade classes. Two of these classes, those of teachers D and M, were selected for the present study, as the differences between these classes were prominent. The mathematical theme of the study is calculating probabilities in 2-dimensional sample space, which are not necessarily equi-probable. We analyzed students' responses on the question in the post-test (Azmon, 2010). The question was: "The spinner on the dial is turned twice. (The partitions on the dial are equal.) What has greater chance, or are the chances equal: The dial will stop twice on the area of 1, or the dial will stop once on the area of 3 and once on the area of 4? Explain." We analyzed the students' responses regarding the correct claim and its correct justification.



In addition, the corresponding whole class discussion concerning this topic in both classes was analyzed, using AiC and DCA. The analyses focused on several variables: numbers of turns (total, teacher and students); numbers of words uttered by teacher and students; the number of cases in which students took the roles of knowledge agents and their followers as an indication for creative ideas raised in the whole class forum; and characterizing both teachers' questions as conceptual or procedural, and also according to other type of talk moves (prompts, evaluation, re-voicing). Note that teachers' prompts have the potential to elicit creative mathematical thinking. All variables were categorized and quantified. Finally, we interpreted the differences between the post-test findings of the two classes on the basis of the findings from the whole class discussions.

FINDINGS

Findings from the post-test

As can be seen from Table 1, there is a difference in students' responses to the post-test question between the two classes. Two thirds of Class M provided the correct claim as compared to only one third of Class D (the chances of having the dial stop twice on the 1 area are not equal to that of having the dial stop once on 3 and once on 4). As for justifications, although for both classes less than half of the students provided a correct justification, the percentage of students from Class M who provided a correct justification was more than three times as high as that of Class D. Moreover, all students of class M provided some justification, whereas 17% of class D did not.

We were puzzled by these results. We know that students from both classes had a good mathematical background, and had good grades in general. We also know that the two teachers, D and M were experienced and considered as good teachers in their school. What may have caused the differences?

% of Class	Students answers			Students justifications		
	No claim	Wrong claim	Correct claim	No justification	Wrong justification	Correct justification
D	19	45	36	17	69	14
M	5	29	66	0	52	48

Table 1: Students responses to the post-test question (percentages)

Findings from the whole class discussions

A first quantification on turns within the whole class discussion in both classes reveals quite similar results concerning the total number of turns; the number of teacher turns and of student turns (first two columns of Table 2). However, when considering the numbers of words uttered by the teacher and students in each class, we start to see a difference between the two whole class discussions: on average, students in class D said much less at each turn than those in class M. Moreover, teacher D spoke five times as many words than all her students, while teacher M spoke less than twice as many words as all her students.

We also analyzed the whole class discussion to identify cases of students acting as knowledge agents where other students followed their ideas. In D's class we found 5 such cases, while in M's class we found 8 cases. We see this as additional evidence to the more active and creative role played by the students in class M as compared to class D.

Class	Teacher turns	Student turns	Teacher words	Students words	Teacher Words per turn	Students words per turn
D	44	43	931	180	21.16	4.19
M	43	42	820	459	19.07	10.93

Table 2 – The number of turns and words of the different categories in the two classes

Further analysis of the teachers' talk, points to some other differences between D and M (Table 3).

Class	Teacher's prompts	Conceptual questions	Procedural questions	Rhetorical questions	Evaluation	Re-voicing
D	12	26	27	14	11	13
M	25	24	12	1	4	15

Table 3 – Number of teacher turns according to different categories

While the two teachers asked almost the same number of conceptual questions, D asked twice as many procedural questions. Also, she asked many rhetorical questions, to which she provided the answer herself. By doing so, she reduced her students' opportunity to participate in the discussion and expressed their creativity. Also, D made three times as many evaluative utterances. This points to an IRE pattern of instruction. As opposed to that, M gave twice as much prompts as D. Prompts may serve to move the discussion to an argumentative mode, and potentially allow for creative ideas to flourish.

To illustrate the whole class discussion in each class we bring here very short – but typical, episodes from each class. We start from Class D.

9 D: Why do you think the game isn't fair? How did you check?
10 Shay: I made a table
11 D: You made a table and multiplied? Can you tell me what the sample space is? How many options are there?

In turn 155 D asked two questions. While the first is conceptual, the second has a procedural flavor, and indeed the student answered procedurally. Turn 157 includes again three questions. The first is re-voicing, the second is conceptual, however the last question is again technical, and the students simply answered the last question. Thus, students who answer such technical questions, have no room to express creative thought. Let us now look at a short episode from class M.

52 M: is the game fair?
53 Students: The game is not fair!
54 M: Not fair because?
55 Student: Because 12 out of 24 is more than 11 out of 24
56 M: So it is really not fair?
57 Itamar: I think the game is fair!
58 M: Because?
59 Itamar: Because 1 out of 24 doesn't change much, 1 out of 24

M [52] asked almost the same question as D [155], but she did not follow it with a procedural question. Also, in turns 54 and 58, M was not satisfied with the claims raised by her students: She prompted them to provide data or warrants for their claims. By doing so, she affordes original responses from her students [Itamar, 59]. Her response is similar whether the student's claim is correct [53] or incorrect [57]. We can see that she seemed to be aware of Itamar's [57] inaccurate claim, but encouraged him to explain [58], providing the whole class an opportunity to listen and consider his reasoning.

DISCUSSION

A gap in students' responses from two classes to the same test caught our attention and led us to closely examine the whole class discussions on the same mathematical issue. And indeed, although the teacher and students uttered the same number of turns, in all other categories we note differences. While in class D the pattern of teachers talk is close to a traditional IRE pattern, In M's class the teacher was able to prompt (Mercer, 2000) her students, providing them with many more opportunities to raise their claims, elaborate them in their own ways, and be creative and active participants. Although we could only present one episode from each class, the difference in depth of students expressing their innovative and diverse ideas between the two whole class-discussions was consistent over all lessons, as can be seen from the quantitative analyses.

To conclude, we were able to point to differences in the teachers' moves in terms of types of questions asked, and in providing prompts to elicit students' creative thinking. Hence, we think that the different genres of whole class discussions brought about by the two teachers explain, at least partially, the difference in post-test responses of students in both classes.

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ALGEBRAIC SYMBOLS AND CREATIVE THINKING

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Abstract. *Simplifying symbolic expressions is usually perceived in middle school algebra as an algorithmic activity, achieved by performing sequences of short drill-and-practice tasks, which have little to do with conceptual learning or with creative mathematical thinking. The aim of this study is to explore possible ways by which ninth-grade students can be encouraged to apply flexible and creative thinking in a context of completing the design of a multiple-choice questionnaire on equivalent algebraic expressions. The findings indicate that students can be engaged in this kind of non-routine tasks and complete it in a satisfactory way. Also, about a third of the participating students were able to display a medium or high level of originality in their construction of equivalent expressions.*

Key words: Creative thinking, equivalent expressions, students as designers, beginning algebra.

INTRODUCTION

One of the challenges faced by beginning algebra students involves mastering procedures related to work with equivalent algebraic expressions (Kieran, 1992). Although today's computerized environments may have decreased the need to master algebraic skills, procedural competence is still a central component in any mathematical activity. However, technological tools shifted the emphasis from performing operations on complex algebraic expressions to understanding their role and meaning. Recently, Friedlander and Arcavi (2012) proposed a conceptual approach, which requires an adoption of some higher-order thinking skills that may foster conceptual understanding of symbolic manipulations. Their list of cognitive processes echoes processes which were frequently investigated in the context of problem solving and creative mathematical thinking (for example, Tabach & Friedlander, 2013) – rather than that of procedural work. Our aim in the present study is to explore the potential of involving students as designers, engaging in creative mathematical thinking in the context of equivalent algebraic expressions.

BACKGROUND

The need to foster creative mathematical thinking in school mathematics nowadays is acknowledged by the mathematical research community (e.g., Leikin & Pitta-Pantazi, 2013) and also, to some extent, by some educational policy documents. For example, the new Israeli national syllabus for elementary school mathematics recommends to involve students in problem solving activities that require "thinking and creativity". This somewhat vague recommendation is not so easy to find, and is even less easy to implement. As noted by Tabach and Friedlander (2013), there is an inherent tension between two seemingly opposing curricular goals: learning procedures and applying algorithmic strategies in routine tasks, on the one hand, and learning concepts and employing more advanced thinking strategies in solving non-routine problems, on the other hand.

Simplifying symbolic expressions is usually perceived in middle school algebra as an algorithmic activity, which in many cases is achieved by performing sequences of short drill-and-practice tasks, which have little to do with conceptual learning or with creative mathematical thinking. Friedlander and Arcavi (2012) proposed an alternative conceptual approach aimed at designing tasks suitable for middle school algebra, aimed to integrate conceptual and procedural learning. Friedlander and Arcavi (2012) based their conceptual approach to the learning and teaching of algebraic procedures on a list of nine cognitive processes – five of which are relevant to the current study and are elaborated next.

Reverse thinking is employed when, in contrast to simple procedural tasks, the "direction" of the activity is reversed, and calls for backward thinking, or for reconstructing a procedure already performed but missing. *Posing* a problem about a given situation can also be considered a reversed non-routine activity that requires creative thinking (Singer, Ellerton, & Cai, Eds., 2013). In the context of equivalent expressions, reverse thinking requires reconstructing expressions or equations based on given parts and/or on the final result of an exercise.

Constructing examples and counterexamples requires students to understand the meaning of a concept or a procedure, to apply reverse thinking and to justify and think creatively. Zaskis and Leikin (2007) consider creating examples and counterexamples an important research tool, as it provides a 'window' into students thinking.

Identifying errors and misconceptions requires students to follow, interpret, and evaluate a solution produced by a real or a fictitious peer. This activity casts students in a "teaching" role and as a result, it may induce reflection about their own or other students' potential sources of difficulties. This ability frequently involves justifying, thinking critically, and analyzing and monitoring results. In this case as well, students are granted the teacher's traditional role to identify errors.

Meaningful application of algebraic operations requires students to work with multiple representations or multiple solution methods or answers and to discuss solution processes (rather than results). Tasks characterized by multiple solution methods or outcomes are considered appropriate for eliciting creative mathematical thinking (Leikin, 2009). Note that a main goal of these tasks is to stimulate observation of, and reflection on- rules of algebraic operations.

Divergent thinking is frequently required by tasks that involve and promote multiple solution methods, a wide variety of answers, meaningful mathematical discussions, and opportunities for creative solutions. Guilford (1967) linked creative thinking with divergent thinking (or production). Divergent thinking involves the creative generation of multiple answers to a particular problem (in contrast to convergent thinking, which aims for a single, correct solution).

The study

The task employed in this study required students to design a multiple choice questionnaire on the topic of equivalent algebraic expressions. In each of the six test items, students were given an algebraic expression (the stem) and were asked to provide several correct answers (equivalent to the given expression), and several incorrect distractors (non-equivalent to the given expression).

In order to provide examples of correct answers, students were expected to be familiar with accepted ways of simplifying expressions, whereas the construction of "good distractors", required awareness to misconceptions and errors in performing operations on algebraic expressions.

We claim that student work on such a *Make-a-Quiz* activity involves each of the five cognitive processes described above:

- The task calls for thinking processes that are reversed to those of a "direct", (*Take-a-Quiz*) test item, that requires to provide a correct simplified expression for a given more complex one.
- Given an algebraic expression, the task requires students to construct examples of equivalent expressions and counterexamples of non-equivalent ones.
- The design of “good” distractors requires both a deeper understanding of procedures of simplifying algebraic expressions, and an awareness of common misconceptions and errors related to these procedures.
- Requiring students to provide more than one equivalent expression and several "good" (i.e., not easily detectable) distractors involves divergent thinking.

METHODOLOGY

The aim of this study is to explore possible ways by which ninth-grade students can be encouraged to apply flexible and creative thinking in a context of completing the design (i.e., providing distractors for given stems) of a multiple-choice questionnaire on equivalent algebraic expressions. The specific questions that we addressed in this study were:

- What is the students' level of originality in constructing expressions that are equivalent to a given one (i.e., in providing correct answers in the context of their designed test)?
- What is the students' level of awareness of potential errors and misconceptions with regard to algebraic procedures, as expressed in constructing expressions that are ***not*** equivalent to a given one (i.e., in providing "reasonable" distractors in the context of their designed test)?
- What is the students' level of originality expressed in reflecting potential errors and misconceptions with regard to algebraic procedures in constructing expressions that are ***not*** equivalent to a given one (i.e., in providing "reasonable" distractors in the context of their designed test)?

A total of 56 students learning in three ninth-grade heterogeneous mathematics classes of one urban middle-grades school participated in the study. Their background in algebra was acquired in a two-year beginning algebra course that included the basic procedures related to simplifying algebraic expressions and solving linear equations.

Two questionnaires were administered within a time frame of one ninety-minute lesson. The aim of the first questionnaire was to acquaint students with the structure of multiple-choice items, with thinking processes involved in answering this type of items and with principles involved in their design. Each of the five items in this *Take-a-Quiz* preparatory activity included an algebraic expressions given as a stem and six additional expressions – some of them provided as correct answers (i.e., expressions equivalent to the stem) and some others provided as distractors (i.e., non-equivalent to the stem expression). This questionnaire required students to mark all the correct answers.

The second *Make-a-Quiz* activity was the research tool of this study. This questionnaire required students to design an algebraic quiz of multiple-choice questions regarding equivalent algebraic expressions.

Its instructions were: "*Write four distractors for each quiz item. Try to give more than one correct answer and some good distractors*", and the following expressions were given to students as the stems of the six test items:

(a) $7-2\cdot(x-3)=$; (b) $5x-2x\cdot(x-3)$; (c) $\frac{x}{2}+2x$; (d) $10-\frac{1}{4}x+\frac{x}{2}$; (e) $1-\frac{x-7}{2}$; and (f) $(5-x)\cdot(6+x)$.

Responses were collected, and analyzed in two groups of correct answers (equivalent expressions) and distractors (student-perceived procedural errors).

The equivalent symbolic expressions were scaled according to three levels of originality:

- low – expressions derived by applying direct algorithmic simplification of the given stem expressions, (for example, by collecting like terms)
- medium – expressions derived by making a relatively slight change (usually based on applying the commutative law or an arithmetical modification) on a simplified expression
- high – expressions derived by making a non-routine (and usually non-algorithmic) change.

Additional aspects of this study include:

- an analysis of the equivalent symbolic expressions according to the method employed in their construction
- an analysis of the non-equivalent expressions according to two categories – (1) expressions based on identifiable errors, and (2) expressions based on unidentified errors.

Due to the limitation of space, these aspects will not be reported here, but will be presented at the conference.

FINDINGS

As 56 students participated in the study, we could expect to have ($56\cdot4=$) 224 responses for each of the six given expressions. However, since in some cases students did not provide all the four required expressions, or repeated twice the same answer, the total number of responses was smaller than the optimal number.

Table 1 provides some basic information on the distribution between equivalent and non-equivalent expressions. For each of the analyzed four items, the number of equivalent expressions provided by students was about a third of the total number of responses. This figure indicates that some students chose to provide more than one equivalent expression per item. Additionally, we would like to note that

- the number of different types of expressions provided by students in an item was not uniform
- the number of different types of expressions provided by students as distractors was considerably larger (both in relative and in absolute numbers) as compared to the corresponding numbers of equivalent expressions.

Stem	No. of responses	Equivalent Expressions		Non-Equivalent Expressions	
		Number (%)	No. of different types	Number (%)	No. of different types
$7-2(x-3) =$	220	70 (32%)	18	150 (68%)	73
$5x-2x(x-3)=$	209	47 (22%)	11	162 (78%)	102
$\frac{x}{2}+2x=$	199	76 (38%)	32	123 (62%)	66
$(5-x)(6+x)=$	204	59 (29%)	14	145 (71%)	95

Table 1: Global analysis of student responses (equivalent and non-equivalent expressions).

As stated in the methodology section, the equivalent expressions given in four of the six items were further analyzed with regard to their level of originality. Table 2 shows some of the more frequent types of responses, and Table 3 shows the distribution of these expressions by level of originality (in percent).

	$7 - 2(x - 3) =$	$5x - 2x(x - 3)=$	$\frac{x}{2}+2x=$	$(5 - x)(6 + x)=$
Low	$7 - 2x + 6$; $13 - 2x$	$5x - 2x^2 + 6x$; $11x - 2x^2$	$2.5x$; $\frac{5x}{2}$	$30 - x - x^2$; $30 + 5x - 6x - x^2$
Medium	$-2x + 13$; $\frac{7}{1}-2\cdot\left(\frac{x}{1}-3\right)$	$(20x - 9x) -2x^2$; $5x + 6x - 2x^2$	$\frac{5x}{10}+\frac{30x}{15}$; $\frac{2x}{4}+\frac{2x}{1}$	$-x^2 - x + 30$; $30 - x^2 - x$
High	$7 + 2(3 - x)$; $2(-x + 3) + 7$	$5x + 2x(3 - x)$; $5x + 2x(-x + 3)$	$\frac{10x}{4}$; $\frac{3x}{2}+x$	$(5\cdot6 + 5x) - (x\cdot6+x\cdot x)$; $(5 - x)6 + (5 - x)x$

Table 2: Some of the more frequent responses distributed by level of originality.

	$7 - 2(x - 3) =$	$5x - 2x(x - 3) =$	$\frac{x}{2}+2x=$	$(5 - x)(6 +x) =$
Low	66	83	63	77
Medium	8	6	22	19
High	22	10	15	8

Table 3: Distribution by level of originality (in percent).

The findings presented in Table 3 show that the majority of responses were categorized as being at a low level of originality, and about a third of student responses were at a medium or high level of originality.

CONCLUSIONS

The purpose of this study was to explore the potential of certain types of short tasks on topics related to algebraic procedures to elicit and possibly foster creative thinking. A *Make a Quiz* design task was administered to 56 ninth-grade students at the beginning of their third year of algebra learning.

Our findings indicate that students can be engaged in this kind of non-routine tasks and complete it in a satisfactory way. Following Sternberg and Lubart's (1999) definition of creativity – an ability to produce unexpected, original, appropriate and useful pieces of work, we can conclude that students could complete the task at various levels of originality.

A large group of students was able to provide equivalent expressions at a low level of originality – involving direct simplification of a given expression. This finding is in line with Ervynck (1991), who suggested that in certain conditions, in order to reach a creative activity, a preliminary technical stage is needed. However, our findings also show that about a third of the participating students were able to display a medium or high level of originality in their construction of equivalent expressions. The variability in levels of originality between different items (i.e., different types of expressions) needs further investigation. Finally, we would like to note that the types of (both equivalent and non-equivalent) expressions provided by the students may have some pedagogical implications both in teaching algebraic procedures and in our efforts to promote creative thinking in daily school activities.

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A STUDY ON THE RELATIONSHIP BETWEEN CREATIVE SCHOOL ENVIRONMENT AND CREATIVE STUDENTS` PERSONALITY, PROCESS AND PRODUCT

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Abstract. *The paper examines the characteristics, which transform the school environment into a creative school environment, devised by lecturers and researchers to stimulate the mathematical creativity of the learners. These characteristics are presented in view of determining whether the creative environment has an impact on developing students' creative characteristics, supporting students' creative process and supporting development of students' creative product.*

Key words: mathematical creativity, gifted students, creative environment, creative student, creative process, creative product

INTRODUCTION

Creativity is a paradoxical construct to study because in many ways it is self-defining (Juter & Sriraman, 2011). Many researchers are looking for its basic elements, in order to create conditions for their stimulation and development, and thus to satisfy the needs of the society of talented people, who will provide new scientific and social achievements.

Creative minds in Mathematics do not follow wide-known methods for solving mathematical problems. Free from any standards and guided by their interests, having developed specific traits of their creative personalities and mastered relevant knowledge, skills and competencies in many subject areas, they discover original methods and create new knowledge in favour of their professional egos and society as a whole.

The characteristic features of creativity can be outlined in four large groups: creative person, creative process, creative product, creative environment (press) (Piriov, 1981; Leikin & Pitta-Pantazi, 2013).

The focus of this paper will be directed to looking for the characteristics, which transform the school environment into a creative school environment, devised by lecturers and researchers to stimulate the mathematical creativity of the learners.

These characteristics are presented in view of determining whether the creative environment has an impact mainly on developing students' creative characteristics, supporting students' creative process and supporting development of students' creative product.

Environment for Developing Students' Creative Characteristics

Professional experience of teachers of Mathematics, reports from organizers of competitions and certain scientific research results show significant correlation between mathematical creativity and achievements of students (Bahar & Maker, 2011; Walia, 2012).
This is the reason, in many countries such as Russia, Romania, China, Bulgaria, etc. to identify gifted/creative students in Mathematics through solving open-ended problems and to organize study environment for advanced and comprehensive study of many mathematical

topics, developing students' problem-solving skills, and participation in mathematics competitions and winning prizes.

Sample teacher training resources for implementing this type of training have been created by a team of 9 countries under the European project MATHEU (Makrides et. al., 2004). They include theoretical mathematical items and problems, methods for solving complicated problems in both primary and secondary education, methods of instruction in Mathematics and identification methods for gifted students. However, many students trained in these schools find their professional and scientific realization in science, technology, engineering and mathematics (Wai et. al., 2010).

Still, some of them remain on the level of maths problem-solvers only. For the sake of solving this problem and generating the experiences from competitions and Olympiads, new math competitions were organized as SUB 14 (Amaral & Carreira, 2012), as well as developing scientific papers, presenting them at scientific students` conferences, solving one common very difficult problem by a group of trainees etc., that involve collaborative and individual work with a mentor.

These competitions foster mathematical creativity of the students because they provide free choice of topic in the field of interests, long periods of time for solving the problems, opportunity for literature research. Thus trainees probe new methods for creating knowledge while experimenting with their own ideas.

The effective learning environment, which has strong impact on the development of the students' mathematical abilities, creative thinking, creative problem-solving and creative learning skills, has to respond to their interest, special educational intellectual, affective and social needs (Stoltz et. al., 2015), has to be challenging, flexible (Rogers, 2007), to provide opportunities for self-work, to fully develop their demonstrated talents, to provide the methodology for creating new knowledge (Velikova, 2002), to provide conditions for learning at their own pace, presenting the results to the appropriate audience and reach reward (Renzulli & Reis, 1997), to be attractive for parents and society as a whole. Davies et. al. (2013) generates these qualities in three broad themes: 1) *physical environment*; 2) *pedagogical environment*; and 3) *role of partnerships beyond the school.*

Therefore, the educational environment is expected to create conditions for interaction with the creative personality of the student including his/her interests, knowledge and skills, aiming at developing his/her creative competencies and mastering new knowledge and skills for collaborative and individual creative mathematical activity.

Environment for Supporting Students' Creative process

The psychological nature of the creative process includes creative and critical thinking (Piske et. al., 2014), insight, intuition and inagination, tightly connected with tasks commitment (Renzulli & Reis, 1997), maths memory (Krutetskij, 1968), durability and others.

The process of supporting, for example, of *the students' scientific imagination* by learners through the learning environment runs in the following stages (Ho et. al., 2013):

1) *Initiation Stage* - stimulating generation of as many ideas as possible; supporting student' curiosity and adventurous spirit; providing students with models from their own life experiences; using the principle "imagine without limitation";

2) *Dynamic Adjustment Stage* - supporting students in finding as many relationships as possible among ideas using *association* and to *transform* or *elaborate* one idea into a novel one; and

3) *Virtual Implementation Stage* - applying the appropriate ideas in solving the problem by conceptualisation, organisation, and formation.

Students' creative process may be supported by implementation of some teaching methods as (Lassig, 2013):

1) *adaptation* of known ideas/methods within a particular area;

2) *transfer* ideas/methods from one mathematical area to another one; new application of an existing product/ idea/method;

3) *synthesis* - combining some ideas and creating something new; and

4) *genesis* - basing creativity on an aggregate of ideas and experiences.

The learning environment influences the creative process by teaching content, methods for solving groups of tasks, methods for creating new knowledge, the attitude of the trainer towards creativity, opportunities for innovation, playfulness, task-oriented, collaborative learning, teacher's guidance, generating new ideas imagination, creative, critical, independent and investigative thinking, risk-taking (Lin, 2011; Piske et. al., 2014). Application of the mathematical model-eliciting activities on the base of non-routine, open-ended real-life challenge mathematical situations and artefacts which include also computers and recent multimedia instruments stimulate the mathematical creative process (Bonotto & Sanot, 2014) and develop students' creative thinking (Eric, 2008), cognitive and affective characteristics such as motivation, interest, self-efficacy and persistence, metacognition and self-reflection (Gilat & Amit, 2014).

The discovery of new knowledge by the students as design of new mathematical problems, new interpretation, new method for transforming a problem from one area of Mathematics to another, etc. (a new mathematical problem, a new interpretation, a new method for transforming a task from one area to another, etc .) is associated with the development of problem-posing abilities, which demand the implementation of three types of learning situations (Stoyanova & Ellerton, 1996): 1) *free situations* (generating problems from a given, contrived or naturalistic situation); 2) *semi-structured situations* (exploring the structure of a given open situation and complete it by applying knowledge, skills, concepts and relationships from their previous mathematical experiences); and 3) *structured situations* (activities are based on a specific problem).

Therefore, the support for mathematical creative process of the students is realized by creating a creative learning environment that stimulates thinking and imagination, the development of problem-posing abilities and mastery of methods for creating knowledge.

Environment for Supporting Development of Students' Creative Product

There are several important characteristics of the students' creative product that distinguish usually the creative product. They are:

1) new to the student who created it;

2) intended for a specific audience and therefore must be presented;

3) new to this audience;

4) valuable for the student himself at the moment of their creation, rather than for the future society.

The creative product can be: new mathematical interaction, task, transformation, the application of an idea already developed in a new way, or finding new application areas for well-known methods and settings (Velikova, 2002).

The Schoolwide Enrichment Model (Renzulli & Reis, 1997), based on the “Three-Ring’’ Conception of Giftedness” of J. Renzulli, presents criteria for assessment of the characteristics of the students’ products. But very often, they have been used by researchers for identifying creative productive gifted students. There is a little research on the characteristics of the mathematical students’ creative products (essays, posed problems and solved problems). Velikova (2005) have presented the experimental results of extracurricular work with 16-18 year old creative students. The creative learning environment is organized to work in three stages: 1) forming a strong interest in students towards a specific mathematical area; 2) preparation for creative work - developing the creative abilities, research skills, acquiring knowledge to a degree of applicability in the field of the student’s interests; acquainting with the methods of creative work in the area; achieving an optimal motivation level of creative work; and 3) joint and independent creative mathematical work in and through which the student expresses himself as a creator. Original hard mathematical problems are designed (geometrical inequalities and equalities) which have been created by applying many different kinds of original geometrical transformations of one triangle to another one.

Therefore, school environment may have an impact on the quality of the students' creative product.

CONCLUSIONS

The experience of the author, as well as the results of many studies, allows us to draw the following conclusions. The quality of the learning environment, the personality of the student, the process of learning and instruction, and the assumed goals outline the profile of the future adult. But the characteristics of the creative learning environment, the creative teacher, the creative process and creative product, each of which is a complex phenomenon, can provide the potential for a successful combination and a specific interaction, and the result is students' creativity and the modeling of the student as a creator in the future.

Media can be used as a creative learning environment, where and through which certain conditions are created for the training of the gifted students to implement a joint and independent creative mathematical activity, such as formation of interest, motivation of the learner for creativity, development of the specific qualities of the creative personality, mastery of new knowledge and scientific methods, formation and development of skills that are directly involved in creative activities in specific scientific field, promotion of creative thinking and its formation process and presentation of the creative product to relevant audience.

As such, creative talent can manifest in certain people (not all) at a certain time (not always), under appropriate conditions (Renzulli and Reis, 1997). Therefore, the role of the teacher is always complex because she/he should aim at creating a learning environment that offers a wide range of options of processes and products to stimulate students` creative capacities.

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MATHEMATICAL CREATIVITY AND THE AFFORDANCES OF DYNAMIC AND INTERACTIVE MATHEMATICS LEARNING ENVIRONMENTS

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Abstract. *The proposal presents our investigation of the opportunities to enhance mathematical creativity in the dynamic and interactive mathematics learning environments (DIMLE). These environments provide affordances for students to learn visually, dynamically, and exploratively. Our analysis of the tasks, which have been created to explore how students benefit from these affordances, reveals that DIMLE may provide opportunities to improve students' mathematical creativity.*

Key words: Dynamic and Interactive Mathematics Learning Environments, Mathematical Creativity, Affordances.

INTRODUCTION

This proposal investigates the possible contribution of the affordances of the Dynamic and Interactive Mathematics Learning Environments (DIMLE) to the raise in mathematical creativity. The following sections provide a brief history of the term DIMLE and its affordances, theoretical considerations in relation to mathematical creativity, and description of research for the context. Then, we explore the affordances of DIMLE through mathematical creativity framework. To our knowledge, this type of exploration will be the first of its kind, therefore, we assume to make a unique contribution to –or at least start a discussion in –the mathematical creativity literature. The research question leading us for this discussion is set as, what opportunities do the DIMLE and their affordances provide for educators to foster mathematical creativity of students?

Dynamic and Interactive Mathematics Learning Environments (DIMLE)

Technological innovations influence mathematics education in a number of ways. Computer Algebra Systems (CAS) such as Maple, Derive, and Mathematica, and dynamic applications such as GeoGebra, Geometer's Sketch Pad, Cabri, and Fathom, are the examples of leading software systems, which have been employed in classrooms (Leikin & Grossman, 2013; Leung, Baccaglini-Frank, & Mariotti, 2013; Masalski & Elliott, 2005; Prusak, Hershkowitz, & Schwartz, 2011; Scher, 2005).

In the literature, the latter group of software systems, i.e., GeoGebra, Geometer's Sketch Pad, and Cabri – is usually referred to as Dynamic Geometry Systems (DGS) or Dynamic Geometry Environments (DGE). However, since we believe that these software systems are suitable for learning not only geometry, but also other disciplines of mathematics, such as algebra and data management, the term Dynamic and Interactive Mathematics Learning Environments (DIMLE) was developed as more appropriate (Martinovic & Karadag, 2010). A detailed discussion on the preference of using DIMLE rather than DGS or DGE is done elsewhere (Martinovic & Karadag, 2012) and falls outside the scope of this paper. The affordances of the DIMLE have been explored in a discussion group meeting at the PME-NA in Reno, US, in 2011 and at the I. IDEAL Conference in Bayburt, Turkey, in 2012. The discussion has revealed that the DIMLE have three major characteristics that the students need to perceive: they provide opportunities for visual learning, dynamic learning, and explorative learning (Karadag, Martinovic, & Freiman, 2011; Martinovic, Karadag, & Freiman, 2012).

The Affordances of DIMLE

The visualization and visual learning have been of interest for both mathematicians and mathematics educators. The visualization may refer to use of graphs, diagrams, and tables, which together with or without algebraic and numerical representations. Visualization allows for concretization of the mathematical concepts either in paper-and-pencil or computer environments (Goldenberg, 1991; Rivera, 2011; Zimmermann & Cunningham, 1991). The process of visualization—for example, drawing a function on any medium—may help one to analyze the properties of concepts, explore the relationships between concepts, such as between the function and its derivative, and proceed to further mathematization processes.

Similarly, dynamism is another feature of DIMLE, which supports cognitive processes alongside with dragging, moving, or changing dynamic elements (Leung, Baccaglini-Frank, & Mariotti, 2013; Leung & Lee, 2013; Pelczer, Singer, & Voica, 2014). For example, one may use a slider to change the values of numerator or denominator of a fraction to understand if by doing so, the fraction gets bigger or smaller. Therefore, it is suggested that dynamic learning is associated with a dynamic change, which may provide an opening for asking "what if" questions, such as "what will happen if I increase this value?", followed by further exploration and making conjectures.

While visual and dynamic learning have stemmed from the features of DIMLE, explorative learning is associated with either visualization or dynamism, and allows learners to create conjectures, test conjectures, or verify ideas—in short— to engage in a mathematical experiment (Karadag & Aktumen, 2013; Leung, Baccaglini-Frank, & Mariotti, 2013).

The Framework of Mathematical Creativity

Creativity may be defined as ability to develop something new, mostly as a result of a cognitive effort. Eyring (1959) states, "[c]reativity is rarely a single flash of intuition; it usually requires sustained analysis of great many observations to separate out the significant factors from the adventitious" (p. 3), while discussing scientific creativity through a chemistry scientist perspective.

The literature suggests fluency, flexibility, and novelty as the traits of the mathematical creativity (Guilford, 1959; Leikin, 2009; Sriraman, 2005). In terms of mathematical tasks, the fluency is defined as the ability to develop a number of possible solutions, while the flexibility could be considered as the ability to adapt to the new situation, when the problem or the task is altered. Although Guilford (1959) suggests four types of fluencies (word, associational, expressional, and ideational fluencies) and two types of flexibilities (spontaneous and adaptive flexibilities), we believe that associational or ideational fluencies and adaptive flexibility could better fit into mathematical context. Novelty—or originality—is the ability to develop new approaches or new solutions, or to transfer an existing solution to the similar cases.

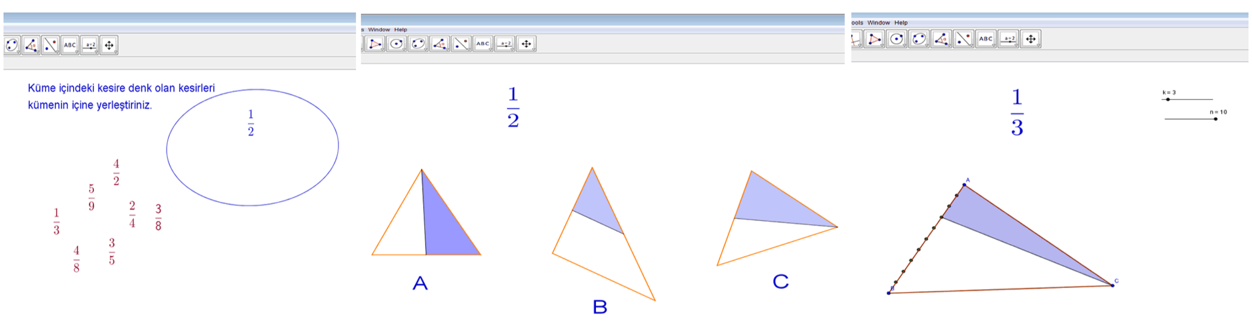
THE CONTEXT

The context for this report is a research project, whose data collection is completed and analysis is in progress. The project is about understanding ways in which students benefit from the affordances of the DIMLE. The data were collected from 14 Grade 6 and 7 students from Bayburt, Turkey. The participant students were provided mathematical tasks to solve in GeoGebra, and Wink, a screen capturing software, was used to record their performance. In addition to screen capturing students' ways of solving mathematical tasks, each session was videotaped and observed by two graduate students. Thus, we have enough documents to triangulate the findings.

The Tasks

The topics covered in data collection were fractions, transformational geometry, 2D and 3D geometry, patterns, and ratio and proportions. This proposal illustrates only the examples from fractions—equivalent fractions and ordering of fractions.

Figure 1 contains screenshots from tasks Fraction 11 (A), Fraction 21 (B), and Fraction 31 (C). The Fraction 11 (A) task asks students to drag the fractions which are equivalent to 1/2 into the set, while Fraction 21 (B) asks them to identify visual representations that are equivalent to ½ and type their answers in a separate document. That is, ideas behind the two tasks are almost the same, and the major difference is in the representation of fractions. The task Fraction 31 (C) is much more dynamic compared to the other two because it asks students to create visual representations of the fraction 1/3 by dragging the sliders (seen on the top right). There are two sliders, one of the sliders controls the total number of points located on the side of the triangle, thus, defining the number of equal partitions of the segment, whereas the other slider determines the number of partitions defining the base of the shaded triangle.



A B C

Figure 4: The screenshots of the tasks related to equivalent fractions

The Figure 2 illustrates screenshots from another three tasks, this time related to ordering fractions. The task Fraction 41 (A) asks the students to put the given fractions in order from smallest (on the left) to the largest (on the right), while the Fraction 42 (B) is a visual representation of the same task. This time the students need to determine which part of the area of the triangle is shaded and to order the resulting fractions in the separate sheet. Again, the Fraction 44 (C) provides a dynamic task, asking students to find the fractions between 1/3 and 3/5 by dragging the sliders (located at the top right); the top slider controls the numerator of the middle fraction whereas the bottom slider controls the denominator of the fraction.

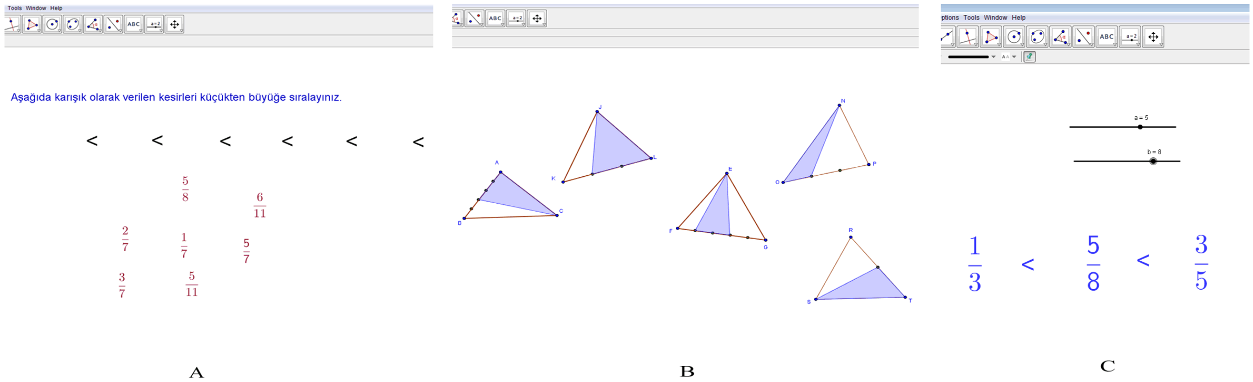


Figure 5: The screenshots of the tasks related to ordering fractions

Analysis of the Tasks

In this section we reflect on the affordances of the DIMLE through the mathematical creativity framework. The framework puts three traits of mathematical creativity forward: fluency, flexibility, and novelty. In our discussion of the DIMLE, we have defined three affordances, namely: visual, dynamic, and explorative. Now, we visit and apply the framework to each of these affordances.

Visualization might help students attend to the tasks and underlying concepts and principles intuitively. Even for other representations used in mathematics—algebraic and numeric—the basic reception channel in majority of cases is visual, if there is no specific reason for the others, and the "creativity is the ability to see (or to be aware) and to respond" (Fromm, 1959, p. 44). Thus, we may assume that visualization might foster creativity to a certain degree, but what about the traits? It is widely recognized that thinking, learning, and communicating through visuals is important skill for the 21st Century learning (as aspects of visual literacy) (Little & Berry, 2012). We may expect students to become more fluent than ever before if they are confronted with more visual representations in mathematics. The current education curriculum delivered in schools seems to hinder students to rely on their visual perception. Depending on how much students rely on their visual perception, there seems to be a possibility to foster their ability for achieving flexibility and novelty as well. They are supposed to be more flexible because they need to move back-and-forth between multiple representations and to transfer mathematical information they draw from one representation to another. Regarding novelty, the possibility of using more visual representations might help teachers and students develop new modes of representations. For example, using areas of triangles to illustrate fractions (as in Figure 2 (B)) seems to be original compared to the traditional depictions through rectangular stripes or pie charts.

Dynamism seems to provide even more opportunities for creativity. For example, students may create a number of variations of the task by dragging the slider, and having all these variations on the screen they may attend to a number of cases. This dragging opportunity may help them think more fluently and flexibly because each time they come across a new case, they need to assess the case and move forward. For example, students working on the Tasks 31 (Figure 1(C)) or 44 (Figure 2(C)) have more chance to see more problem cases than the students working with traditional textbook exercises. The possibility to help students find original approaches to the existing problems may be assumed high if a relevant task is designed.

One may confidently assume that the opportunity to explore mathematics in DIMLE enhances all three traits of mathematical creativity, but the novelty most of all. If the task is to explore; the students need to act fluently and flexibly, and have more chance to develop novel products. Our observations of students during highly exploratory tasks demonstrate them going through the state of "emotional ambivalence"—simultaneously experiencing positive and negative emotions—the state that is positively correlated with creative productivity, such as finding unusual relationships between concepts (Fong, 2006).

DISCUSSION

This proposal presented our effort to investigate the research question, *what opportunities do the DIMLE and its affordances provide for educators to foster mathematical creativity of students?* Our humble analysis of tasks has revealed that the DIMLE provide opportunities to enhance the traits of mathematical creativity. Particularly, the dynamic and explorative affordances of DIMLE seem to have highest potential. Nickerson's (1999) claim that, "[students] also need opportunities to make real discoveries – to learn from personal experience" (p. 416) supports this notion, because the DIMLE offer students opportunities for experimenting with mathematics tasks dynamically and for exploring their underlying concepts and principles.

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WORKSHOPS

RATIONAL CREATIVITY: ALGORITHMS OR INNOVATION? TEACHING OPERATIONS WITH FRACTIONS

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Abstract. *In the United States, operations with fractions are most frequently taught by asking students to memorize standard algorithms and perhaps later apply these to word problems. To counteract this trend, in this workshop, participants will actively and creatively explore problems involving operations with fractions from a variety of online and curricular resources, beginning with contextual situations, making sense of operations, and using a variety of models and techniques to solve and extend the problems.*

Key words: mathematical creativity, operations with fractions

RATIONALE AND PURPOSE

Research has shown that not only are fractions poorly understood by students and adults alike in the United States, but that this difficulty is often a roadblock to higher-level mathematics as well as many careers and even everyday adult life. When I was a student, I learned to divide fractions using the saying, "Ours is not to reason why, just invert and multiply." Today we can find several online versions of ways to memorize without thinking, including videos of rappers saying, "Keep, change, flip", meaning that to divide fractions, you keep the first fraction the same, change the sign, and flip the second fraction. In this workshop, we will investigate a variety of resources and techniques to help students develop deep and enduring understanding of operations with fractions as well as creative habits of mind, setting a foundation for later mathematical expertise, enjoyment and innovation.

MATHEMATICAL TOPIC

The emphasis in this session will be on making sense of operations with fractions using a variety of innovative methods and resources. Methods will include the Standards for Mathematical Practice from the Common Core State Standards (CCSS); problem solving, reasoning, constructing arguments, modeling, using tools, looking for structure, and expressing regularity in repeated reasoning, as well as a suggested additional standard for mathematical practice on creativity and innovation, "Solve problems in novel ways, and pose new mathematical questions of interest to investigate" from *Using the Common Core State Standards for Mathematics with Gifted and Advanced Learners,* a monograph co-published by the National Association for Gifted Children (NAGC), the National Council of Teachers of Mathematics (NCTM) and the National Council of Supervisors of Mathematics (NCSM). For example, instead of asking students find the quotient for ½ ÷ 3/6, ask students how many different equations they can write using four different one-digit whole numbers for a, b, c, and d where $a/b \div c/d = 1$ and prove that they have found all possible solutions. Extend this by asking students how they might find the largest (or smallest) possible quotient using four different one-digit whole numbers for a, b, c, and d for the expression $a/b \div c/d$. Ask students to explore more deeply by writing their own extension questions, perhaps by changing the operation, changing the parameters for the numbers, or asking students to write situations where these equations can be used to solve the problem.

For example, the CCSS includes tables showing at least a dozen distinct types of contextual problems for addition and subtraction and another dozen distinct types for multiplication and division. Examples in the CCSS are only given for small whole numbers, however, and teachers and students alike have difficulty in applying these ideas to fractions or other rational numbers. For example, can you write two different types of problems for $1\frac{3}{4} \div \frac{2}{3}$? One situation should involve equal groups with the group size unknown. (How many in one group?) The other situation should involve equal groups with the number of groups unknown. (How many groups?) Student solutions to these will be shared, including the use of models such as bar diagrams and number lines.

The website, Math at the Core (www.pbslearningmedia.org/collection/**mathcore/**) , from the U. S. Public Broadcast System (PBS) Learning Media has a variety of resources for middle school mathematics, including videos, games, riddles and other interactive materials related to operations with fractions. In this workshop, we will explore some of these, as well as a variety of fraction problems from the award-winning, Javits grant-supported *Project M³: Mentoring Mathematical Minds* for advanced elementary students and the research- and CCSS-based middle grades program, *Math Innovations*.

PARTICIPANT INVOLVEMENT

Participants will be actively engaged in solving and posing a variety of fraction problems using an open-ended problem-solving/problem-posing heuristic and assessment rubric. Participants will learn innovative questioning strategies based on a series of "who, what, when, where, why, and how" questions that encourage students to delve deeply into engaging complex problems and persevere in their solutions using a variety of creative strategies and models, forming generalizations that lead to proofs and rules, and posing additional questions.

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Math Innovations (Grades 6-8): www.kendallhunt.com/mathinnovations/

Math at the Core: Middle Grades: www.pbslearningmedia.org/collection/mathcore/

Project M³: Mentoring Mathematical Minds (Grades 3-5): www.projectm3.org

EFFECTIVE FEEDBACK FOR EFFICIENT LEARNING: A COMPUTER-BASED SYSTEM OF ASSESSMENT

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Abstract. *This workshop will present the facilities offered by a computer-based system of assessment (the STARs technology), and will involve the participants in challenging working-group activities concerning: designing an assessment item, proposing significant distracters, giving adequate feedback, and developing relevant hints to guide the student in the problem-solving process.*

Key words: multiple-choice problems, computer-based assessment, distractors, feedback.

INTRODUCTION

Within a large research program, Lakoff and his colleagues explained us how people built conceptual thinking and high-order thinking skills by moving from concrete physical activities through metaphors (Lakoff, 1987; Lakoff & Nunez, 2000). Within the World Wide Web, information tends to leave its usual containers, such as books, moving the pleasure of reading far from mind-and-body interaction. This is a process of abstraction that affects our way of thinking. While supports are disappearing and distances become irrelevant for communication, collaborative systems become equally more feasible and more needed.

On the other hand, the Web 2.0-practices embody a dynamic and participatory view of knowledge that generates though problems of information overload and the need for critical assessment skills (Singer & Singer, 2010). In the context of the knowledge economy, the web-based educational resources have tremendously extended. They are based on the view of knowledge as a collective social product, and thus the access tends to be open. The importance of open educational resources has been widely recognized in the learning community (e.g. Downes, 2007).

Taking into account these aspects, a computer-based system of assessment that allows the development of collaborative databases of items has been built in order to offer students appropriate questions and personalized feedback to challenge learning with understanding. More specific, the STARs technology offers students a criteria analysis of their answers, with the purpose to help them progressing by their own learning rhythm and style. The purpose of this workshop is two folded: familiarizing the participants with the system facilities, and involving them into the specific design of assessment items.

COMPUTER-BASED ASSESSMENT

The STARs technology involves developing a database of items, some of which are those proposed in the Kangaroo contests. Within the Kangaroo competition, the proposed problems are of multiple-choice type. Most of these problems are insight problems that can be solved quickly, but they require a certain intuition to discriminate between the answer variants. In principle, when designing multiple-choice questions, distracters (wrong answers) could be chosen at random, around the correct answer.

However, an appropriate choice of these distracters is useful not only for the development of intuition in solving, but especially to capture the passage between successive phases of the problem-solving process (Singer & Voica, 2013).

Therefore, a first phase in the development of this type of items is that, while generating the problem statement, to identify significant distracters that can give indications about the students' possible mistakes. In order to orient students' further learning, each distracter is accompanied by a commentary addressed to the student, which explains him/her the error done, the possible source of this error, and possible strategies (i.e. new items to solve) to overcome it.

A second phase of conceiving such questions is the formulation of a relevant hint for solving: the way these hints are formulated can have a deep impact on the solver (Pelczer, Singer, & Voica, 2011). The challenge here is how to keep a balance between initiating the solving process and uncovering the main idea of the solving, which should be still discovered by the student in a learning approach.

WORKSHOP ORGANIZATION

The workshop will be organized in three sections.

In the first section, we will present the facilities offered by the computer-based system and we will show concrete examples of how a student gets personalized feedback.

In the second part of the workshop, participants will split in 3-4 working groups and they will be engaged in challenging activities concerning the design of an item. More specifically, each group will receive the text of a problem, and the participants will propose significant distracters, comments to a student who (wrongly) choose those distracters, and will discuss relevant hints able to guide the students in the problem-solving process.

In the third part of the workshop, the working groups will present their proposals and we will share together educational issues related to the presented assessment system.

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VISUALIZING GEOMETRIC CONCEPTS USING A NOVEL TYPE OF 3D PUZZLES

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Abstract. *A novel type of 3D puzzles is investigated as a manipulative tool in teaching geometry. Concepts from mathematics and adjacent sciences: complementarity, tessellation, interlocking and kinetics are illustrated by physically constructing modular structures of Archimedean polyhedron. We have experimented with this educational resource in class, with 2nd and 5th graders and found out that it improves spatial intelligence and reasoning. The workshop introduces a new component of this tool, using flexibly connected interchangeable modules, and engages participants in hands-on activities similarly to those used in class.*

Key words: XColony Knowledge Discovery Kit, concrete manipulatives, mathematical creativity.

INTRODUCTION

The idea that promoting and developing creative mathematical thinking in school becomes a necessity is widely accepted by mathematics community (e.g. Leikin & Pitta-Pantazi, 2013). The use of manipulatives is view as a possibility to develop students' creativity (Berk, 1999). Therefore, as teachers, we are interested in including manipulatives in mathematics classrooms activities, to promote understanding and foster creativity.

The goal of this workshop is to introduce to the math teaching community new project-oriented activities that could be used for teaching geometric creativity and spatial education to students in middle school.

In the workshop, we describe our experience in using XColony Knowledge Discovery Kit (KDK). KDK is designed starting from the polyhedral computing system XColony (Alexe, 2013), described more detailed at http://www.xcolony.eu/.

Starting from planar figures obtained from regular hexagons, we can obtain T-modules (tetrahedron based), O-modules (octahedron based), and I-modules (icosahedron based); using these elementary modules, one can obtain complex buildings, as presented in Fig. 1.

Previous research show that KDK enhance students' strategic thinking and team-work skills (Alexe, Voica, & Voica, 2013; Alexe, Alexe, Voica, & Voica, 2014).

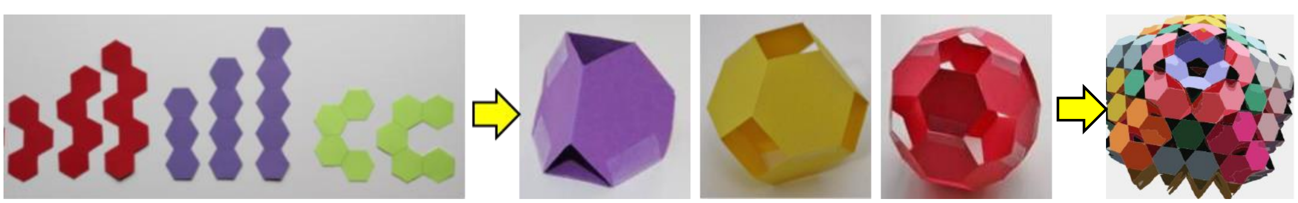

Figure 1. The elementary modules T, O, and I and complex XColony buildings

THE EDUCATIONAL PROGRAM

The KDK-based educational program was applied in *Herăstrău Middle School* - Bucharest, Romania, during the first semester of the scholar year 2014-2015, as part of optional classes offered through the school curriculum. We used the XColony KDK one hour per week, with 33 second grade students (8-9 years old) and 29 fifth grade students (11-12 years old).

The main objectives of this program, from the students' perspective, were: identifying geometric and continuing patterns; developing communication abilities in describing geometrical configurations; developing spatial intuition and mathematical creativity.

Using control-groups, we found that XColony KDK activities enhance creativity and improve the communication skills of students (Alexe, Alexe, Voica, & Voica, 2015).

WORKSHOP ORGANIZATION

The workshop is organized in three sections.

The first section introduces XColony KDK concepts and provides examples of activities that were carried out by our students. A visual introduction and step-by-step instructions are presented through short videos.

The second section proposes hands-on activities in groups, asking participants to build some XColony constructions. Each group formulates the strategy to construct the final object and carries out the activities as planned.

After building some simple constructions and connecting them into more complex ones, participants are challenged to explore this geometric environment, to investigate how to connect these modules flexibly, and to experiment various transformations and kinetic properties.

The **third section** is focused on discussing educational aspects in organizing class activities based on XColony KDK.

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ACTIVITIES THAT ENGAGE GIFTED AND TALENTED STUDENTS IN PRODUCTIVE STRUGGLES WITH DESIRABLE DIFFICULTIES: A MODEL ENCOURAGING INTENSE DISCOURSE AND DEEP REASONING

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Abstract. *We can't help our students to be seekers if we aren't seekers ourselves. In this research-based, practice-oriented workshop we explore the benefits of productive struggle by creating desirable difficulties that help gifted and talented students shake up either hastily-assumed conclusions or loose thinking and to construct "new" knowledge by focusing attention on the transfer of related knowledge to new situations.*

Key words: *problem solving; reasoning; discourse; productive struggle; intrinsic motivation.*

Not every gifted and talented student wants to be a mathematician, but every student can experience thinking like one. It is important to realize that being good at mathematics is not evidenced by how many answers you know. Instead, being good at mathematics is evidenced by what you do when you don't know the answer. We must help gifted and talented students, in fact, all students, to construct their own "new" knowledge, and, through modeling, apply that knowledge in ways different from the situation in which it was learned (National Council of Teachers of Mathematics, 2000, 2007, 2010).

Sometimes learners express a reluctance to look at mathematics in an alternative way to their initial exposure to the topic (Rohrer, 2009), (Rohrer & Taylor, 2006). Pleas of "You're going to confuse me!" may signal an unrecognized confusion that is ALREADY present (Kornell & Bjork, 2008). Using the principle that "No matter what IT is, the chances of finding IT are dramatically increased if you're looking for IT" we will explore techniques that encourage and reinforce the use of modeling as a means of nurturing thinking and reasoning (Lester, 1994).

The ideas we gather are like so many pieces of colored glass at the end of a kaleidoscope. They may form a pattern, but if you want something new, different, and beautiful, you'll have to give them a twist or two. You experiment with a variety of approaches. You follow your intuition. You rearrange things, look at them backwards, and turn them upside down. You ask "what if" questions and look for hidden analogies. You may even break the rules or create new ones. All this leads to new levels of confidence in one's thinking (Polya, 1945).

There are many benefits to be gained by gifted and talented students through the use of productive struggle with desirable difficulties that are designed to encourage reasoning as well as enhance both long-term retention and transfer of learning (Bjork & Bjork, 2011). Out of apparent chaos and confusion, the effective use of productive struggle can bring about the emergence of a deeper understanding and appreciation of the world we encounter and interpret every day (Epp, 2004).

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USE OF ALGEBRAIC REASONING FOR EARLY IDENTIFICATION OF MATHEMATICAL TALENT

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Abstract. *The purpose of this workshop is to explore the thinking processes of upper-elementary mathematically promising students (MPS) and the strategies in which they execute generalization processes in their algebra tasks. It also offers a set of non-routine and cognitively challenging algebra tasks in related problems. Participants will also engage with upper-elementary mathematically promising students' work and create some extended related problems. The workshop will conclude with an overall discussion of the MPS algebraic thinking characterizations the workshop's implications for teaching and research.*

Key words: Mathematically promising students, algebraic thinking, and upper elementary students.

INTRODUCTION

Are we underestimating the level of mathematical ability and interest of many of our upper-elementary mathematically promising students when they involve algebra tasks? The term *mathematically promising* is defined by The National Council of Teachers of Mathematics (NCTM) task force (1994) as "the students who have been traditionally identified as gifted, talented, precocious and students who have been traditionally excluded from rich mathematical opportunities" (p. 310). Often considered to be in the top 3 to 5 percent on standardized tests in math, the NCTM broadened the definition to include ability, motivation, belief, experience and opportunity. Teachers must be able to identify the mathematical talents of gifted students and differentiate their instructions. They should view themselves as math "scouts" on the lookout for new talent and potential, and utilize multiple ways of identifying these students under a broad definition of what defines strong math aptitude. Teachers must have special expertise in mathematics, reasoning, and problem solving, and an understanding of the cognitive, emotional, and behavioral development of school-aged students. (Greenes, Teuscher& Regis, 2010).

Gavin (2011) shared Piaget observation on students' mathematical work. Based on his observation, he categorized students with mathematical talent into three types: (a) thinking and acting very abstractly (algebraic cast of mind), (b) having high ability in spatial visualization and reasoning (geometric mind), and (c) students who demonstrate a combination of both (a) and (b). In addition to these, some other mathematics skills (e.g., speed, computation, formula memorization, etc.) are not necessary requirements for math aptitude (Krutetskii, 1976). Some people have natural mathematical abilities, while others acquire it over time. Krutetskii argued that while actual mathematical abilities are not innate, some children will become more able than others through subsequent development and experiences.

Attributes of the mathematically promising students

Krutetskii (1976) developed a checklist of the key elements of mathematical thinking observed in the mathematically promising students:

1. An ability to generalize and formalize mathematical material: (a) to see what is common in what seems evidently different, (b) to isolate form from content to abstract and operate with formal structure, (c) to operate with numerals and other symbols.
2. An ability to shorten the reasoning process: (a) to see logical reasoning (proof), (b) to reverse a mental process.
3. Flexibility of thought**:** (a) to switch from one mental operation to another, (b) to use non-standard approaches to algebra problems.
4. An ability with spatial concept reasoning.

THIS WORKSHOP

This workshop chose algebraic thinking in a particular generalization (i.e., functional thinking) and emphasizes its unique contribution to the mathematical development of promising students specifically. The tasks used numerical, visual, and spatial patterns and showed a great potential for revealing and creating algebraic knowledge (Amit &Neria, 2008).

I will be examining the generalization methods of capable upper elementary students when solving linear and nonlinear pattern and generalization problems. This investigation will discuss and define the various stages of students' generalizations and the use of representations through justification methods. Mathematically promising students' use of near and far generalization techniques will be elaborated on the solution strategies that led to successful problem solving. The final section will focus on challenging the participants to discover and investigate generalization problems through actively engaging various forms of algebraic tasks. The workshop will conclude with an overall discussion of MPS's algebraic thinking characterizations and the workshop's possible implications for teaching and research.

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SYMPOSIA

THE NATURE OF CREATIVITY

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Abstract. *Symposium on the Nature of Creativity is proposed in the light of the often met assertion "there is no single, authoritative perspective or definition of creativity" leaving practitioners without a clear and supportive viewpoint as conceptualization of creative learning varies due to the diversity of the proposed definitions of creativity.*

Key words: Creativity, bisociation, Aha! moment, affect, habit

We are proposing a Symposium: The Nature of Creativity in mathematics education. We are inviting colleagues working with different approaches to creativity to the Symposium in order to contrast and compare these approaches with the hope of arriving at a common understanding of creativity. That hope is based upon the developmental theory of scientific thinking proposed by Diana Kuhn of Teachers College (Kuhn, 1999).

The arrival at the joint point of view on creativity is urgent at present since there is no single, authoritative perspective or definition of creativity (Leikin, 2011, Kattou et al., 2011) leaving practitioners without a clear and supportive viewpoint as conceptualization of creative learning varies due to the diversity of the proposed definitions of creativity (Kattou et al., 2011). At the same time creativity in mathematics has been proposed as a very effective motivational tool for students (Prabhu, 2015).

There are indications that the new definition of creativity proposed by (Prabhu and Czarnocha, 2014) on the basis of the bisociation theory of Arthur Koestler's Act of Creation may introduce the cohesive element into the field by addressing many of its present ambiguities. Bisociation, or Eureka experience is "*the spontaneous leap of insight…which connects previously unconnected matrices of experience* [*frames of reference*] *and makes us experience reality on several planes at ones*" (p.45). This definition is complemented by Koestler's recognition that *the act of creation is the act of liberation_– it's the defeat of habit by originality!*" (p.96). Thus bisociation plays here a dual role, that of a cognitive reorganizer and that of an effective liberator from a habit - it's planting a double root for creativity. These two observations are central, in our opinion, for the exploration of its nature. The focus of the Act of Creation Theory is on the bisociative leap of insight, that is, an Aha! moment, or a moment of understanding, — a phenomenon that contains both an affective component of the 'Aha' moment and a cognitive component of the synthesis of two previous unrelated matrices of thoughts the hidden analogy as Koestler would refer to it. These components in so far as they can be observed amongst the general population suggests Koestler's framework as suitable for measuring and analysing the creative aspect of self discovery during learning.

The Symposium will reflect upon and seek answers to some of the following sets of issues/questions:

- What is Creativity both intuitively and in accordance to definitions in the field representing understanding of the community?

- What is the nature of the Aha moment? How does "connecting two separate frames" of thinking or a discourse impact the development of a relevant mathematical schema of a learner? Is Aha moment primarily an affective experience as Liljedahl, (2009) suggests or does it represent the cognitive/affective duality proposed by Czarnocha, (2014)? What is the role of the "simultaneity of attention" (Baker, Czarnocha, 2015) during the Aha moment as well as during its investigation by researchers?
- What can be the utility of Creativity in mathematics education? To what degree facilitation of creativity in a "regular" classroom of mathematics can motivate students towards the subject? What are the effective methods of such facilitation? What does democratization of creativity mean?
- How can we measure the depth and the scope of the cognitive development occasioned by the Aha moment, both theoretically and empirically? Proposals by Campbell et al, (2012) and Baker (2015) will be considered.

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FOSTERING MATHEMATICAL GIFTEDNESS IN REGULAR CLASSROOM SETTING AND IN SPECIAL PROGRAMS

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Abstract. *The aim of this symposium is to offer participants an opportunity to exchange views on questions about fostering mathematical gifted students. Research about the role of expertise and deliberate practice underlines actual opinions about giftedness as a result of developmental processes (Ericsson, 2006).*

Key words: actiotope model, deliberate practice, expertise, fostering giftedness

QUESTIONS

What are the essentials of the process which comes along with the development of competences? Starting with an example based on the actiotope model of giftedness (Phillipson & Sun, 2009; Ziegler, 2004; Ziegler & Phillipson, 2012) the participants are invited to discuss the essential aspects of developing mathematical competencies. Because this model takes into account the broadness of influencing factors and their interplay, restrictions for fruitful discussions are necessary. The focus can be laid on the role of teacher, on the kind of problems, on questions about fostering conditions f. e..

Among the relevant questions to be addressed are the following:

I.

- How can teachers provide appropriate learning environments for developing mathematical capabilities in the whole classroom settings?
- Are there differences between the approaches in regular classrooms and pull out programs?

II.

- What are the characteristics of problems suitable for challenging gifted students?
- Can these kinds of problems be used in regular classroom settings?
- Should problems be posed as complex problems or should special cognitive components of problem solving be trained isolated?

III.

What are the opportunities:

- of competitions?
- of learning environments based on natural differentiation?
- of progressive research problems?

During the session, participants will be presented with the results of a case study about one problem used in regular classroom as well as in a drop out program (Pamperien, 2008).

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